LPTENS-99/54
hep–th/9912089
December 1999

Integrable structures in classical off-shell 10D supersymmetric Yang-Mills theory

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Abstract

The physical content of the integrable modification of super Yang-Mills theory in ten dimensions put forward earlier (hep-th/9811108) is investigated. To this end, group-algebraic methods are developed which allow to derive the set of physical fields and their equations of motion from the superfield expansion of the supercurl, systematically.

The Lax representation underlying this system is based on a one parameter set of light rays. Its $SO(2,1)$ covariance is made manifest and acts by Möbius transformations on the spectral parameter. After appropriate dimensional reduction, this allows a consistent projection of the spectrum to $SO(2,1)$ singlets. We analyze in detail the system after reduction to seven dimensions. It describes the interactions of two $SO(7)$ vector fields – one being the original Yang-Mills vector potential –, and two antisymmetric $SO(7)$ tensor fields.

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[1]Supported by EU contract ERBFMRX-CT96-0012.

[2]UMR 8548: Unité Mixte du Centre National de la Recherche Scientifique, et de l'École Normale Supérieure.

Contents

1 Introduction.

Recently, progress was made in applying exact integration methods to supersymmetric Yang-Mills theory in ten dimensions. The starting point were the flatness conditions in superspace which have been known for some time to be equivalent to the field equations [2, 3]. It was shown in [1], that there exists an on-shell light cone gauge, where the superfields may be entirely expressed in terms of a scalar superfield satisfying two sets of equations. The first is linear and a general solution was derived [1]; the second is similar to Yang's equations and has been handled by methods similar to earlier studies of self-dual Yang-Mills in four dimensions. A general class of exact solutions[3] has been obtained [1] and a Bäcklund transformation put forward [6].

So far, however, it has not been possible to simultaneously solve the two sets of equations. Only a particular class of solutions of the Yang type equations has been found, which is not general enough to solve the other (linear) set. Returning to a general gauge, one may see that deriving the scalar superfield satisfying the linear subset of equations is equivalent [6] to solving a particular symmetrized form of the flatness conditions. This symmetrized form was shown to be explicitly integrable directly, since it arises [4, 5] as compatibility condition of a Lax representation, similar to the one of Belavin and Zakharov [7], which may be handled by the same powerful techniques as in the case of self-dual Yang-Mills in four dimensions.

The original Lax representation [3] is formulated for light-like rays in ten dimensions, which play the role of spectral parameters. As such, this representation has not been very useful in practice, and the recent progress corresponds to retaining only a subset of the flatness conditions – and hence to modify the dynamics – in such a way that the techniques of [7] become applicable.

Concretely, we consider a subset of Lax equations with just a single spectral parameter. Geometrically, this corresponds to restricting the Lax connection to a one parameter set of light rays. In section 3, this aspect is analyzed in more detail. The simplest non-trivial situation corresponds to a one parameter family of light rays whose spatial components vary in a fixed plane. We show that, with a particular choice (the $\{x^9, x^8\}-$plane in our

[3]keeping, however, only the dependence upon one time and one space coordinates in contrast with the dimensional reduction which will be discussed below. This is probably not essential.

conventions), this gives back the symmetrized flatness conditions mentioned above. In this paper, we study the physical content of the integrable system associated with this particular choice of plane.

To this end, we employ an important new ingredient, namely, the covariance of this Lax pair under the $SO(2,1)$ subgroup that acts in the $\{x^0, x^9, x^8\}$−plane, provided that the spectral parameter simultaneously undergoes a Möbius transformation. Thus, the breaking of the original Lorentz invariance in ten dimensions, which is the key point of the present approach corresponds to $SO(9,1) \to SO(2,1) \otimes SO(7)$. For Weyl-Majorana spinors, this leads to the separation

$$\mathbf{16} \to \mathbf{8} + \mathbf{8} \ , \tag{1.1}$$

(where the r.h.s. denotes a pair of $SO(7)$ spinor representations which form a doublet under the $SO(2,1)$). This is instrumental in defining the involution which is the key to applying methods modeled over the self-duality requirement in four dimensions; here, it involves the exchange of the two spinor representations. At the end of section 3, we show, how the $SO(2,1)$ covariance of the Lax representations allows to consistently truncate the spectrum of physical fields described by our modified Yang-Mills dynamics, by retaining singlets only. Effectively, this corresponds to a dimensional reduction to seven dimensions.

In the rest of the paper we temporarily forget about the Lax representation, and concentrate upon deriving the field content and dynamical equations contained in the symmetrized flatness conditions. In general, the superspace Lax representations are such that the dynamical equations appear as conditions on the fermionic components of the field strength. This is true for the original ten-dimensional Yang-Mills equations (equivalent to vanishing of the full supercurvature), as well as in our case. Since the symmetrized flatness conditions are weaker than the full ones, we effectively go partially off-shell. In particular, this gives rise to the appearance of more physical fields appearing in the superfield components and a modification of the dynamics. At this point, it is worth recalling that, in the standard treatment [8], the method used to eliminate the unphysical components of the superfields makes an essential use of the equations of motion. It is thus not applicable to our case. The main aim of the remaining part of present work is to devise a more direct and general method, which is also applicable to our modified equations.

Section 4 is devoted to a systematic study of the expansion of the superfield equations in powers of the odd variables θ. We derive recursion relations with an interesting structure. The elimination of unphysical components of superfields is done recursively and involves two operators noted S and T. The first satisfies a simple quadratic equation while the second is nilpotent. Thus our equations bear some analogy with the descent equations [9]. In this approach, the field equations are enforced by further applying a projector K, and we thus study the interplay between S, T, and K on general ground. Applying this method to the symmetrized flatness conditions, in section 5, we explicitly identify the physical field content, in the case where no fermionic fields are present. Section 6 then is devoted to deriving the field equations. To a large extent, we concentrate upon the case of $SO(2,1)$ singlets, for which the field spectrum is shown to be still remarkably simple. Some concluding remarks are given at the end.

2 Superfield conventions.

The notations are essentially the same as in the previous references. The physical fields are noted as follows: $X_a(\underline{x})$ is the vector potential, $\phi^\alpha(\underline{x})$ is the Majorana-Weyl spinor. Both are matrices in the adjoint representation of the gauge group $\mathbf{G}$. Latin indices $a = 0, \dots, 9$ describe Minkowski components, Greek indices $\alpha = 1, \dots, 16$ denote spinor components. We use the Dirac matrices

$$\Gamma^a = \begin{pmatrix} 0_{16\times16} & \Big((\sigma^a)^{\alpha\beta}\Big) \\ \Big((\sigma^a)_{\alpha\beta}\Big) & 0_{16\times16} \end{pmatrix}, \quad \Gamma^{11} = \begin{pmatrix} 1_{16\times16} & 0 \\ 0 & -1_{16\times16} \end{pmatrix}. \tag{2.1}$$

We will use the superspace formulation with 16 odd coordinates θ^α. The general superfield expansions of a superfield $\Phi\,(\underline{x}, \underline{\theta})$ may be written as

$$\Phi(\underline{x}, \underline{\theta}) \;=\; \sum_{p=0}^{16} \Phi^{[p]}(\underline{x},\, \underline{\theta}) \;\equiv\; \sum_{p=0}^{16} \sum_{\alpha_1,\dots,\alpha_p} \frac{\theta^{\alpha_1}\cdots\theta^{\alpha_p}}{p!}\, \Phi^{[p]}_{\alpha_1\dots\alpha_p}(\underline{x})\,. \tag{2.2}$$

The grading is given by the operator

$$\mathcal{R} = \theta^\alpha \partial_\alpha\,, \qquad \Big[\,\mathcal{R}\,, \Phi^{[p]}\,\Big] = p\,\Phi^{[p]}\,. \tag{2.3}$$

Superderivatives are defined by

$$D_\alpha = \partial_\alpha - \theta^\beta\,(\sigma^a)_{\alpha\beta}\,\partial_a\,, \quad \text{such that} \quad [D_\alpha\,, D_\beta]_+ = -2\,(\sigma^a)_{\alpha\beta}\,\partial_a\,. \tag{2.4}$$

The odd super vector potential, valued in the adjoint representation of the gauge group, is denoted by $A_\alpha\,(\underline{x},\underline{\theta})$. We define its supercurvature as

$$\begin{aligned} F_{\alpha\beta} &\equiv D_\alpha A_\beta + D_\beta A_\alpha + [A_\alpha,\, A_\beta]_+ + 2\,(\sigma^a)_{\alpha\beta}\,A_a\,, & (2.5)\\ \text{with} \qquad A_a &\equiv -\frac{1}{16}\,(\sigma_a)^{\alpha\beta}\,\left(D_\alpha A_\beta + A_\alpha\,A_\beta\right) & (2.6)\end{aligned}$$

This defines the even vector potential $A_a\,(\underline{x},\underline{\theta})$ as function of A_α.[4] The superfield formalism is invariant under gauge transformations

$$\begin{aligned} A_\alpha &\mapsto g^{-1} A_\alpha\,g + g^{-1} D_\alpha g \\ F_{\alpha\beta} &\mapsto g^{-1} F_{\alpha\beta}\,g \\ A_a &\mapsto g^{-1} A_a\,g + g^{-1}\partial_a g \end{aligned} \tag{2.7}$$

with an even superfield $g\,(\underline{x},\underline{\theta})$ as gauge parameter. Imposing the so-called transverse gauge condition

$$\theta^\alpha A_\alpha = 0 \tag{2.8}$$

restricts the freedom (2.7) to ordinary gauge freedom, i.e. to gauge parameters g with $[\mathcal{R}, g] = 0$. It has been shown [2, 3, 8] that the transverse gauge condition together with the flatness conditions

$$F_{\alpha\beta} = 0\,, \tag{2.9}$$

implies the Yang-Mills equations of motion

$$\begin{aligned} \partial^a F_{ab} + [\,A_a\,, F_{ab}\,] &= \frac{1}{2}\,(\sigma_b)_{\alpha\beta}\,\left[\,\chi^\alpha\,, \chi^\beta\,\right] \\ (\sigma^a)_{\alpha\beta}\,\left(\partial_a \chi^\beta + \left[\,A_a\,, \chi^\beta\,\right]\right) &= 0\,, \end{aligned} \tag{2.10}$$

for the superfields A_a and χ^α, the latter being defined as $\chi^\alpha \equiv (\sigma^a)^{\alpha\beta}\,F_{a\beta}$ with the curvature

$$F_{a\beta} \equiv \left(D_\beta A_a - \partial_a A_\beta + [\,A_\beta\,, A_a\,]\,\right)\,. \tag{2.11}$$

[4]In the terminology of [10], we have hence resolved the "conventional constraint" $(\sigma_a)^{\alpha\beta} F_{\alpha\beta} = 0$.

In particular, the zero order components of these superfields

$$X_a \equiv A_a^{[0]} \,, \qquad \phi^\alpha \equiv \chi^{\alpha\,[0]} \,, \tag{2.12}$$

satisfy the Yang-Mills field equations . Moreover, (2.9) together with (2.8) gives a unique recurrent prescription of the higher order superfield components in A_a, χ^α and A_α as functions of the zero order contributions, i.e. as functions of the physical fields.

3 The geometry of flatness conditions.

Recall next that the flatness conditions (2.9) have a Lax type representation in superspace [3]. They imply the existence of a $\mathbf{G}$-valued superfield $\Psi[\underline{\ell}]$ for any light-like ten-dimensional vector $\underline{\ell}$, which is defined by the linear system

$$\ell^a \, (\sigma_a)^{\alpha\beta} \, \{D_\beta + A_\beta\} \; \Psi[\underline{\ell}] \;\; = \;\; 0 \,, \tag{3.1}$$

$$\ell^a \, \{\partial_a - A_a\} \; \Psi[\underline{\ell}] \;\; = \;\; 0 \,. \tag{3.2}$$

In turn, the compatibility conditions of (3.1), (3.2) imply (2.9). Clearly, these equations are invariant under multiplication by an overall constant, so that Ψ only depends upon the light-like ray considered.

3.1 A particular representation

An essential step will be to break the $SO(9,1)$ of the original theory. We do so in a particular way, where – as a matter of choice – the coordinates $\{x^0, x^9, x^8\}$ play a special role. Then the simplest formulae will result from the following particular realization of σ-matrices:

$$\begin{aligned}
\left((\sigma^9)^{\alpha\beta}\right) \;&=\; \left((\sigma^9)_{\alpha\beta}\right) \;=\; \begin{pmatrix} -1_{8\times 8} & 0_{8\times 8} \\ 0_{8\times 8} & 1_{8\times 8} \end{pmatrix} \,, \\
\left((\sigma^0)^{\alpha\beta}\right) \;&=\; -\left((\sigma^0)_{\alpha\beta}\right) = \begin{pmatrix} 1_{8\times 8} & 0_{8\times 8} \\ 0_{8\times 8} & 1_{8\times 8} \end{pmatrix} \,, \\
\left((\sigma^i)^{\alpha\beta}\right) \;&=\; \left((\sigma^i)_{\alpha\beta}\right) = \begin{pmatrix} 0 & \gamma^i_{\mu,\bar\nu} \\ \left(\gamma^{i\,T}\right)_{\nu,\bar\mu} & 0 \end{pmatrix} , \quad i = 1, \ldots 8 \,,
\end{aligned} \tag{3.3}$$

$$\gamma^i \gamma^{jT} + \gamma^j \gamma^{iT} = 2\delta^{ij}, \quad i,\, j = 1, \ldots, 8. \tag{3.4}$$

Our index convention is as follows: Greek letters from the beginning of the alphabet run from 1 to 16, letters from the middle of the alphabet from 1 to 8, denoting the two spinor representations of $SO(8)$. Likewise, Roman indices from the beginning of the alphabet run from 0 to 9 whereas Roman indices from the middle of alphabet range between 1 and 8 (later on between 1 and 7):

Using these explicit realizations the flatness conditions (2.9) take the form

$$
\begin{aligned}
D_\mu A_\nu + D_\nu A_\mu + [A_\mu,\, A_\nu]_+ &= 2\delta_{\mu\nu}\,(A_0 + A_9)\,, \\
D_{\overline{\mu}} A_{\overline{\nu}} + D_{\overline{\nu}} A_{\overline{\mu}} + [A_{\overline{\mu}},\, A_{\overline{\nu}}]_+ &= 2\delta_{\overline{\mu}\overline{\nu}}\,(A_0 - A_9)\,, \\
D_\mu A_{\overline{\nu}} + D_{\overline{\nu}} A_\mu + [A_\mu,\, A_{\overline{\nu}}]_+ &= -2\sum_{i=1}^{8} A_i \gamma^i_{\mu,\overline{\nu}}\,.
\end{aligned}
\tag{3.5}
$$

Equations (3.1), (3.2) may be considered as a Lax representation of the field equations (2.10) where $\underline{\ell}$ plays the role of the spectral parameter. As they stand, however, they are not very useful in practice. In the present approach, one weakens the flatness conditions such that they become derivable from a Lax representation where the spectral parameter is a number λ instead of a light ray, and methods inspired from ref.[7] become applicable. Starting from equations (3.1), (3.2), this geometrically corresponds to keeping only the flatness conditions associated with a particular one parameter set $\underline{\ell}(\lambda)$ of light-like rays. Let us discuss such curves on the light cone systematically. First, it is clear that their space part $\vec{\ell}(\lambda)$ cannot point in a fixed direction, since otherwise the corresponding light ray would not depend upon λ and the scalar spectral parameter would drop out, completely. The simplest non trivial case hence is, to assume that the space part $\vec{\ell}(\lambda)$ describes a planar curve.

The set of flatness conditions equivalent to the full Yang-Mills equations may then be obtained as follows. Assuming that $\ell^9 \neq 0$, we may rescale $\underline{\ell}$ so that $\sum_{i=1}^{8} \ell_i^2 = 1$, and introduce the following complex parametrization

$$
\ell^\pm = \pm i\,\frac{1 \mp \lambda}{1 \pm \lambda}\,, \quad \ell^i = v^i\,, \quad \text{for} \quad i = 1, \ldots, 8\,, \quad \ell^\pm \equiv \ell^0 \pm \ell^9\,, \tag{3.6}
$$

with an eight-dimensional unit vector v^i. With the explicit realization (3.3) one then obtains the following form of the linear system

$$
\left\{ \Delta^{\vec{v}}_\mu + B^{\vec{v}}_\mu + \lambda \left(\overline{\Delta}^{\vec{v}}_\mu + \overline{B}^{\vec{v}}_\mu \right) \right\} \Psi[\lambda, \vec{v}] = 0 \tag{3.7}
$$

with

$$\Delta_\mu^{\vec{v}} = D_\mu + i\vec{v}\cdot\vec{\gamma}_{\mu\overline{\rho}} D_{\overline{\rho}} \ , \quad \overline{\Delta}_\mu^{\vec{v}} = D_\mu - i\vec{v}\cdot\vec{\gamma}_{\mu\overline{\rho}} D_{\overline{\rho}} \ , \tag{3.8}$$

$$B_\mu^{\vec{v}} = A_\mu + i\vec{v}\cdot\vec{\gamma}_{\mu\overline{\rho}} A_{\overline{\rho}} \ , \quad \overline{B}_\mu^{\vec{v}} = A_\mu - i\vec{v}\cdot\vec{\gamma}_{\mu\overline{\rho}} A_{\overline{\rho}} \ . \tag{3.9}$$

One may verify, that the full flatness conditions (3.5) on the superfield A_α are recovered as follows. Choose a set of eight vectors $\vec{v}_{(k)}$ such that $v^i_{(k)} = \delta^i_k$, $i, k = 1, \cdots, 8$. Then, imposing the set of compatibility equations

$$\left[\left\{\Delta_\mu^{\vec{v}} + B_\mu^{\vec{v}} + \lambda\left(\overline{\Delta}_\mu^{\vec{v}} + \overline{B}_\mu^{\vec{v}}\right)\right\}, \left\{\Delta_\nu^{\vec{v}} + B_\nu^{\vec{v}} + \lambda\left(\overline{\Delta}_\nu^{\vec{v}} + \overline{B}_\nu^{\vec{v}}\right)\right\}\right]_+ \Psi[\lambda, \vec{v}] = 0$$

for all μ, ν and unit vectors $\vec{v}_{(k)}$, is equivalent to (3.5).

For each fixed $\vec{v}$, the linear system (3.7), (3.8) is similar to the one proposed by Belavin and Zakharov for four-dimensional self-dual Yang-Mills theory [7]. The role of the involution which in that case describes self-duality is played by exchanging the two $SO(8)$ spinor representations by means of $\vec{v}\cdot\vec{\gamma}_{\mu\overline{\rho}}$, here. These equations are straightforward generalizations of the ones already proposed [4] (which correspond to choosing $\vec{v} = \vec{v}_{(8)}$ along the eighth direction). In general, for a given choice of $\vec{v}$, the system (3.7), (3.8) may be solved starting from an ansatz which is meromorphic in λ. This leads to purely algebraic equations – coming from the fact that the bracket in equation (3.7) is linear in λ – which may be solved in essentially the same way as was done for self-dual Yang-Mills. It seems however impossible to carry out the next step and solve the system for all $\vec{v}$, which would really give a solution of the full Yang-Mills equations in ten dimensions. Indeed, equation (3.9) implies that the bracket in (3.7) should be linear in $\vec{v}.\vec{\gamma}$, a very strong requirement, which to satisfy there seems to exists no systematic method.

3.2 Integrable subsector

We thus stick to the earlier proposal to consider the integrable system obtained by solving (3.7) for a particular choice of $\vec{v}$. With $v^i = \delta^i_8$, it is convenient to choose $(\gamma^8)_{\mu\overline{\nu}} = \delta_{\mu\overline{\nu}}$. It then follows from the Dirac algebra (3.4) that the matrices γ^i, $i = 1, \ldots, 7$ are antisymmetric. From now on, Roman indices $i, j, \ldots$ from the middle of the alphabet will exclusively denote the coordinates $1, \ldots, 7$.

The truncation $v^i = \delta^i_8$ gives back the Lax equations with scalar spectral parameter λ, proposed earlier [5]:

$$\mathbf{D}_\mu \Psi[\lambda] \;=\; \mathbf{D}\,\Psi[\lambda] \;=\; 0\,, \tag{3.10}$$

with

$$\begin{aligned}
\mathbf{D}_\mu &\equiv \Delta_\mu + B_\mu + \lambda(\overline{\Delta}_\mu + \overline{B}_\mu)\,,\\
\mathbf{D} &\equiv (1{-}\lambda)^2\,(\partial_+{+}A_+) - (1{+}\lambda)^2(\partial_-{+}A_-) + i\,(\lambda^2{-}1)\,(\partial_8{+}A_8)\,,\\
\Delta_\mu &= D_\mu + iD_{\overline{\mu}}\,, \quad \overline{\Delta}_\mu \;=\; D_\mu - iD_{\overline{\mu}}\,,\\
B_\mu &= A_\mu + iA_{\overline{\mu}}\,, \quad \overline{B}_\mu \;=\; A_\mu - iA_{\overline{\mu}}\,.
\end{aligned}$$

This Lax connection satisfies the algebra

$$\begin{aligned}
[\,\mathbf{D}_\mu\,,\mathbf{D}_\nu\,]_+ &= (1{+}\lambda)^2\,F_{\mu\nu} - (1{-}\lambda)^2\,F_{\overline{\mu\nu}} + i(1{-}\lambda^2)\,(F_{\mu\overline{\nu}} + F_{\nu\overline{\mu}}) - 4\,\mathbf{D}\,,\\
[\,\mathbf{D}\,,\mathbf{D}_\mu\,] &= (1{+}\lambda)^3\,F_{-\mu} - i\,(1{-}\lambda)^3\,F_{+\overline{\mu}} - (1{-}\lambda)^2(1{+}\lambda)\,(F_{8\overline{\mu}} + F_{+\mu}) +\\
&\quad + i\,(1{+}\lambda)^2(1{-}\lambda)\,(F_{8\mu} + F_{-\overline{\mu}})\,.
\end{aligned}$$

The integrability conditions implied by this Lax connection thus are

$$\begin{aligned}
F_{\mu\nu} &= F_{\overline{\mu\nu}} \;=\; F_{\mu\overline{\nu}} + F_{\nu\overline{\mu}} \;=\; 0,\\
F_{-\mu} &= F_{+\overline{\mu}} \;=\; F_{8\mu} + F_{-\overline{\mu}} \;=\; F_{8\overline{\mu}} + F_{+\mu} \;=\; 0\,,
\end{aligned} \tag{3.11}$$

where the second line is a consequence of the first line and the definitions (2.6), (2.11). The constraints (3.11) on the supercurvature are a symmetrized version of (3.5). It was shown earlier [1, 4, 5, 6] that techniques similar to the ones developed for self-dual Yang-Mills in four dimensions are at work. Thus, we find a system of superspace constraints which is completely integrable. In terms of the physical fields encoded in the superfield components this means that going (partially) off-shell, we find completely integrable dynamical equations. This dynamical content of the integrable superspace constraints has to be extracted in the following.

3.3 The $SO(2,1)$ covariance.

It is useful to display the full symmetry of the Lax representation (3.10). To this end, we recall that its origin was the truncation of (3.1), (3.2) to light-like vectors in the $\{x^0, x^9, x^8\}-$plane. With the explicit realization (3.3), these vectors may be written up to an overall factor as

$$\ell^a \sigma_a \;\propto\; \begin{pmatrix} y \cdot 1_{8\times 8} & 1_{8\times 8} \\ 1_{8\times 8} & y^{-1} \cdot 1_{8\times 8} \end{pmatrix} \;\equiv\; \sigma(y) \;, \qquad \text{with} \quad y = i\,\frac{1-\lambda}{1+\lambda} \;,$$

Clearly, any Lorentz transformation in the $\{x^0, x^9, x^8\}-$ plane is realized, up to an overall factor by a change in y. Explicitly, to first order we have

$$\begin{aligned} \sigma(y+\epsilon) &= \left(1+\epsilon\left(1-y^{-1}\right)\right)\sigma(y) + \epsilon\left\{\tau_+\sigma(y) + \sigma(y)\tau_-\right\} \;, \\ \sigma(y(1+\epsilon)) &= \sigma(y) + \epsilon\left[\tau_3,\, \sigma(y)\right]_+ \;, \\ \sigma(y-\epsilon y^2) &= \left(1+\epsilon(1-y)\right)\sigma(y) + \epsilon\left\{\tau_-\sigma(y) + \sigma(y)\tau_+\right\} , \end{aligned}$$

where τ denote the usual Pauli matrices. Thus, the $SO(2,1)$ transformations are realized by Möbius transformations on y. Let us summarize the last equations as

$$\sigma(y + \epsilon\delta_k y) = (1+\epsilon f_k)\,\sigma(y) + \epsilon\left(J_k^T\sigma(y) + \sigma(y)J_k\right) , \qquad (3.12)$$

where by $(\delta_{-1}, \delta_0, \delta_1)$, we denote the generators of $SO(2,1)$, f_k is a number and the J_k are 2 by 2 matrices, implementing the vector representation of $SO(2,1)$ on $\sigma(y)$. Consider the Lax representation (3.10). It may be written equivalently as

$$\sigma\,(y)\,(D+A)\,\Psi\,(y) = 0,$$

where $(D+A)\,\Psi$ is considered as a two-component vector. The situation now simplifies strongly under dimensional reduction. Dropping the dependence on the space-time variables in an appropriate way, it is straightforward to deduce from equations (3.12) that

$$\sigma(y)\left\{D + A\,(\theta + \epsilon\delta_k\theta) + \epsilon J_k A\right\}\Psi\,(y+\epsilon\delta_k y,\, \theta+\epsilon\delta_k\theta) = 0, \qquad (3.13)$$

with $\delta_k\theta = \theta J_k$. More precisely, dropping the dependence on the coordinates $x^\pm$ yields the covariance (3.13) for $k=0$, whereas further dropping the dependence on x^8 ensures (3.13) for all three values of k. In both cases, this implies invariance of the system under

$$\delta_k A = J_k A + (J_k)_\rho^{\;\sigma}\,\theta^\rho\partial_\sigma A \;, \quad \delta_k\Psi = (\delta_k y)\,\partial_y\Psi + (J_k)_\rho^{\;\sigma}\,\theta^\rho\partial_\sigma\Psi \;. \qquad (3.14)$$

for the corresponding values of k.

3.4 Reduction of the dynamics to singlets.

The basic problem of weakening the field equations – i.e. going at least partially off-shell – is, that one encounters a plethora of additional physical fields with complicated tensorial structures. The present $SO(2,1)$ covariance provides a way to organize the spectrum. In particular, upon dimensional reduction to eight and seven dimensions, respectively, we may consistently truncate the spectrum employing the invariance (3.14), discussed above. The crucial point is, that the transformation of y may be algebraically realized on the level of the solution Ψ of the Lax representation, once we assume that Ψ is meromorphic in y.

3.4.1 The 8D reduction

Let us drop the dependence on the coordinates $x^{\pm}$ and further assume that $\delta_0 A = 0$. This corresponds to truncating the spectrum to fields with vanishing z-component of the $SO(2,1)$ spin. It follows from the above discussion that $\Psi[\lambda]$ and $(1+\epsilon\delta_0)\,\Psi[\lambda]$ are both solution of the Lax equations. We may hence consistently impose

$$\delta_0\,\Psi[\lambda] \;=\; 0\;, \tag{3.15}$$

where δ_0 acts according to (3.14). More precisely, this condition reads

$$\left\{\theta^{\mu}\partial_{\mu} - \theta^{\overline{\mu}}\partial_{\overline{\mu}} + (1-\lambda^2)\partial_{\lambda}\right\}\;\Psi[\lambda] \;=\; 0, \tag{3.16}$$

and thus implies the following decomposition of the solution of the Lax pair

$$\Psi[\lambda] \;=\; \sum_{p=0}^{16}\sum_{r=-p}^{p}\left(\frac{1-\lambda}{1+\lambda}\right)^{r}\;\Psi^{[p,r]}(\underline{x},\underline{\theta})\;, \tag{3.17}$$

$$\text{with} \qquad \left\{\theta^{\mu}\partial_{\mu} - \theta^{\overline{\mu}}\partial_{\overline{\mu}}\right\}\;\Psi^{[p,r]}(\underline{x},\underline{\theta}) \;=\; r\,\Psi^{[p,r]}(\underline{x},\underline{\theta})\;,$$

3.4.2 The 7D reduction

Further assuming no coordinate dependence upon x^8, as well as $\delta_{\pm 1}A = 0$, the solution of the Lax representation (3.10) then likewise reduces to a singlet

$$\delta_k\,\Psi[\lambda] \;=\; 0\;, \qquad \text{for} \quad k=-1,0,1\;. \tag{3.18}$$

This implies the decomposition

$$\Psi[\lambda] \;=\; \sum_{p=0}^{16} \sum_{r=-p}^{p} \left(\frac{1-\lambda}{1+\lambda}\right)^{r} \Psi^{[p,r]}(\underline{x},\underline{\theta}) \;, \tag{3.19}$$

$$\begin{aligned}
\text{with} \qquad \left\{\theta^{\mu}\partial_{\mu} - \theta^{\overline{\mu}}\partial_{\overline{\mu}}\right\} \Psi^{[p,r]}(\underline{x},\underline{\theta}) &= r\, \Psi^{[p,r]}(\underline{x},\underline{\theta}) \;, \\
\left\{\theta^{\overline{\mu}}\partial_{\mu}\right\} \Psi^{[p,r]}(\underline{x},\underline{\theta}) &= \Psi^{[p,r+1]}(\underline{x},\underline{\theta}) \;, \\
\left\{\theta^{\mu}\partial_{\overline{\mu}}\right\} \Psi^{[p,r]}(\underline{x},\underline{\theta}) &= \Psi^{[p,r-1]}(\underline{x},\underline{\theta}) \;.
\end{aligned}$$

This shows e.g. that the condition of absence of higher order poles in the spectral parameter – which is the starting point in the derivation of the instanton solutions of four-dimensional selfdual Yang-Mills theory [7] – immediately implies truncation of the superfield expansion of Ψ and furthermore induce certain algebraic relations on the highest order components of Ψ. Both equations (3.17), and (3.19) are compatible with the meromorphic ansatz in λ which is instrumental in the integration technique of [7]. Of course there remains to show that equations (3.10) do have solutions of the form (3.17) or (3.19). We leave this problem for the future, and concentrate in the following upon deriving the dynamical content of the modified field equations. The presented discussion shows, that with the appropriate dimensional reduction, the entire analysis may be consistently reduced to $SO(2,1)$ singlets.

4 Systematics of the supercurl expansion.

The level structure of the superfields has been discussed in detail in [8] (and likewise in [11] for the reduction to $N=3$ supersymmetric Yang-Mills theory in four dimensions). However, this discussion makes an essential use of the Yang-Mills field equations, and thus does not apply to our case. The main purpose of the present paper is to present an alternative method. As central object we consider the modified supercurvature or supercurl

$$M_{\alpha\beta} \;\equiv\; D_{\alpha}A_{\beta} + D_{\beta}A_{\alpha} + \left[A_{\alpha}, A_{\beta}\right]_{+} \;, \tag{4.1}$$

which transforms non-covariantly under the gauge transformations (2.7). It contains the super field strength $F_{\alpha\beta}$ as gauge covariant part

$$F_{\alpha\beta} \;=\; M_{\alpha\beta} - \frac{1}{16}\,(\sigma^a)_{\alpha\beta}\,(\sigma_a)^{\gamma\delta}\,M_{\gamma\delta}\;.$$

Note, that the last relation is not invertible. For later convenience we introduce the general space of superfields symmetric in two additional spinor indices $(\alpha\beta)$ which we denote by $\mathcal{M}$. This space carries the grading (2.2):

$$\mathcal{M} \;=\; \bigoplus_{p=0}^{16}\mathcal{M}^{[p]} \;\equiv\; \bigoplus_{p=0}^{16}\left\{v_{(\alpha\beta),[\gamma_1\ldots\gamma_p]}\,\theta^{\gamma_1}\ldots\theta^{\gamma_p}\right\}\;. \tag{4.2}$$

These superfields have the general decomposition according to $(\mathbf{16}\times\mathbf{16})_s = \mathbf{10}+\mathbf{126}$:

$$M_{\alpha\beta} \;=\; -2\,(\sigma^a)_{\alpha\beta}\,A_a + \tfrac{1}{5!}\,(\sigma^{a_1,\ldots a_5})_{\alpha\beta}\,B_{a_1\ldots a_5}\;, \tag{4.3}$$

$$\text{with selfdual}\quad B_{a_1\ldots a_5} = -\tfrac{1}{5!}\,\epsilon_{a_1\ldots a_{10}}\,B^{a_6\ldots a_{10}}\;.$$

4.1 Algebraic structure of the transverse gauge condition

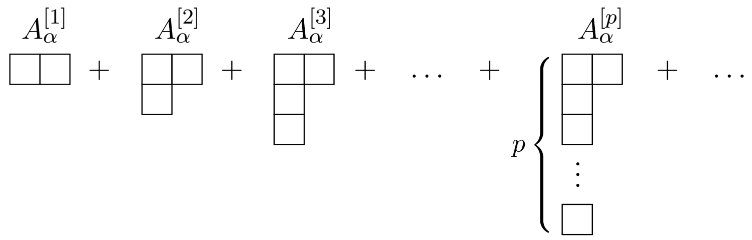
If we do not impose any constraint on the superfields, we simply have to take into account the fact that the gauge freedom (2.7) has been fixed by the transverse gauge condition (2.8) to gauge parameters which do not have higher order superfield components. For the components of the super vector potential this implies

$$\theta^\alpha A_\alpha = 0 \qquad\Longleftrightarrow\qquad A^{[p]}_{[\alpha,\gamma_1\ldots\gamma_p]} = 0\;, \tag{4.4}$$

which is still invariant under ordinary gauge transformations.

This shows that the independent components in the superfield A_α are given by the following sum of Young tableaux[5] for the spinor representation $\mathbf{16}$ of $SO(9,1)$

[5] Here, and in the following, the notion of Young tableaux always refers to (anti)symmetrizing the factors of a tensor product $V^{\otimes N}$ of a given representation V, i.e. always to the Young tableaux of the corresponding permutation group $\mathfrak{S}_N$.



(4.5)

4.1.1 Recurrence relations

Since later on we are going to study the field equations implied by further constraining the supercurl $M_{\alpha\beta}$, we first identify the remaining independent field components in $M_{\alpha\beta}$ after imposing the transverse gauge condition. Equation (4.4) yields

$$(1+\mathcal{R})\, A_\alpha = \theta^\beta M_{\alpha\beta} \qquad \Longleftrightarrow \qquad A_\alpha^{[p+1]} = \frac{1}{p+2}\,\theta^\gamma M_{\alpha\gamma}^{[p]} \tag{4.6}$$

At order p we get from (4.1)

$$\begin{aligned} M_{\alpha\beta}^{[p]} &= \partial_\alpha A_\beta^{[p+1]} + \partial_\beta A_\alpha^{[p+1]} \\ &\quad - (\sigma^m)_{\alpha\gamma}\,\theta^\gamma \partial_m A_\beta^{[p-1]} - (\sigma^m)_{\beta\gamma}\,\theta^\gamma \partial_m A_\alpha^{[p-1]} + \sum_{q=1}^{p-1} \left[A_\alpha^{[q]},\, A_\beta^{[p-q]} \right]_+ \end{aligned}$$

Using (4.6) we may then re-express this relation entirely in terms of $M_{\alpha\beta}$. It is convenient to write it in the form

$$(S+\mathcal{R})\ M + \frac{\mathcal{R}+2}{\mathcal{R}}\, T\, M \ = \ \mathcal{C}\ , \tag{4.7}$$

where we have introduced two linear operators on the set of symmetric superfields, by $(SM)_{\alpha\beta} = S_{\alpha\beta}^{\alpha'\beta'} M_{\alpha'\beta'}$, and $(TM)_{\alpha\beta} = T_{\alpha\beta}^{\alpha'\beta'} M_{\alpha'\beta'}$, with

$$S_{\alpha\beta}^{\alpha'\beta'} \ = \ \delta_\alpha^{\alpha'} \theta^{\beta'} \partial_\beta + \delta_\beta^{\beta'} \theta^{\alpha'} \partial_\alpha \tag{4.8}$$

$$T_{\alpha\beta}^{\alpha'\beta'} \ = \ \theta^\gamma \left(\theta^{\beta'} (\sigma^a)_{\beta\gamma}\, \delta_\alpha^{\alpha'} + \theta^{\alpha'} (\sigma^a)_{\alpha\gamma}\, \delta_\beta^{\beta'} \right) \partial_a \tag{4.9}$$

Moreover, the non linear term $\mathcal{C}$ is given by

$$\mathcal{C}^{[p]}_{\alpha\beta} = (p+2)\sum_{q=1}^{p-1} \frac{\left[\theta^\gamma M^{[q-1]}_{\alpha\gamma},\, \theta^\delta M^{[p-q-1]}_{\beta\delta}\right]_+}{(q+1)\,(p-q+1)}\ . \tag{4.10}$$

Note, that the operator S commutes with $\mathcal{R}$ whereas T raises the level by 2. Thus, (4.7) indeed builds a recursive system, relating the higher levels of $M_{\alpha\beta}$ to the image of the lower ones under T.

4.1.2 Algebraic properties of S and T

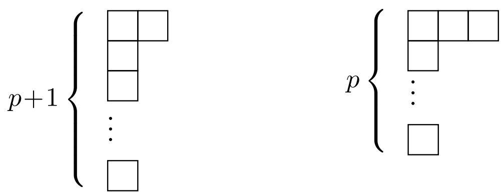
By explicit computation one verifies that the operator S satisfies the equation

$$(S-2)\,(S+\mathcal{R}) = 0 \tag{4.11}$$

Thus, at a given level $\mathcal{R} = p$, the operator S has only two different eigenvalues. We may hence decompose $\mathcal{M}$ into the eigenspaces of S:

$$\mathcal{M} = \mathcal{M}^+ + \mathcal{M}^-\ , \qquad (S+\mathcal{R})\ \mathcal{M}^+ = 0\ , \quad (S-2)\,\mathcal{M}^- = 0\ . \tag{4.12}$$

With (4.8) one finds that $(S-2)$ and $(S+\mathcal{R})$ are the projectors onto the following Young tableaux



$\mathcal{M}^{+\,[p]}$ $\qquad\qquad$ $\mathcal{M}^{-\,[p]}$ $\qquad$ (4.13)

Moreover, one may verify the algebraic relations

$$T^2 = 0\ , \tag{4.14}$$

$$(S-2)\,T \ = \ 0 \ = \ T\,(S+\mathcal{R})\ ; \tag{4.15}$$

i.e. the level raising operator T is nilpotent and acts nontrivially only between $\mathcal{M}^+$ and $\mathcal{M}^-$:

$$T: \mathcal{M}^{+\,[p]} \to \mathcal{M}^{-\,[p+2]}\,. \tag{4.16}$$

The non linear terms of the field equations (4.7) are lumped into $\mathcal{C}$. In the weak field approximation the right hand side of this equation is negligible. Since T is nilpotent, there is then an interesting analogy between equation (4.7) and the descent equations [9]. However, these involve in general two nilpotent operators, whereas in our case S satisfies equation (4.11) instead of being nilpotent.

4.1.3 General solution

Let us separate the two eigenvalues of S in equation (4.7) according to (4.12). It is easy to verify that $\mathcal{C}^+ = 0$. Thus one gets

$$TM^- = 0, \qquad \mathcal{R}M^- + TM^+ = \mathcal{C}.$$

The first relation is automatically satisfied because of (4.15). In conclusion, M^+ is arbitrary, and

$$M^- \;=\; \mathcal{R}^{-1}\left(\mathcal{C} - TM^+\right) \tag{4.17}$$

Thus, M^+ contains the independent components in $M_{\alpha\beta}$ left over by the gauge fixing (4.4). Comparing the Young tableaux (4.13) to (4.5) we hence recover the independent components identified in the vector potential A_α after imposing the transverse gauge. The total and the independent number of components of $M^{[p]}$ are respectively given by

$$\dim \mathcal{M}^{[p]} = 136\binom{16}{p}, \qquad \dim \mathcal{M}^{+\,[p]} = (p+1)\binom{17}{p+2}. \tag{4.18}$$

We give a computation of these numbers in appendix A. Altogether, $M_{\alpha\beta}$ contains 983041 independent components. Since the gauge fixing (4.4) is defined by covariant constraints on the superfield, these components are necessarily expressible in terms of representations of the supersymmetry algebra (2.4). Of course, supersymmetry is not realized level by level. Decomposing M^+ into $SO(9)$ multiplets, we find the following structure

$$\begin{aligned} \sum_p M^{+[p]} &= 983041 \qquad (4.19)\\ &= 1 + (44+84+128)\times(9+16+36+126+128+231+\\ &\qquad +432+576+594+768+924)\,. \end{aligned}$$

The $256 = (44 + 84 + 128)$ corresponds precisely to the smallest irreducible off-shell multiplet of the 10d supersymmetry algebra (2.4) [12]. Consistently, M^+ forms a multiple of this multiplet. The additional singlet in (4.19) corresponds to the fact that we have not fixed the ordinary gauge invariance.

4.1.4 Dual space

For future use, let us recall that we can introduce the dual space $\mathcal{M}_{\rm dual}$ of superfields by means of the bilinear form

$$\langle F|G\rangle = \int d\underline{\theta} \sum_{\alpha\beta} \widetilde{F}^{\alpha\beta} (\underline{x}, \underline{\theta})\, G_{\alpha\beta} (\underline{x}, \underline{\theta}) \tag{4.20}$$

where

$$|G\rangle \in \mathcal{M} \;, \quad \langle F| \in \mathcal{M}_{\rm dual} \;,$$

$$\widetilde{F}^{[p]\,\alpha\beta}{}_{\alpha_1\cdots\alpha_p} \;=\; \frac{1}{(16-p)!} \sum_{\alpha_{p+1},\cdots,\alpha_{16}} \epsilon_{\alpha_1\cdots\alpha_{16}}\, F^{[16-p]\,\alpha\beta\,,\,\alpha_{p+1}\cdots\alpha_{16}}$$

Breaking the $O(9,1)$ invariance, one may identify $\mathcal{M}$ and $\mathcal{M}_{\rm dual}$ by means of $\sigma^0_{\alpha\beta}$, for example. With respect to the decomposition $SO(9,1) \to SO(2,1) \times SO(7)$, the bilinear form (4.20) then yields an $SO(7)$ invariant scalar product on $\mathcal{M}$, on which the $SO(2,1)$ generators act as

$$\delta_0^{\rm ad} = \delta_0 \;, \quad \delta_{\pm1}^{\rm ad} = \delta_{\mp1} \;.$$

We are going to use this scalar product in the subsequent analysis of the superfield constraint (3.11). Note finally, that with respect to this scalar product the operator S from (4.8) is self-adjoint

$$S^{\rm ad} = S \;. \tag{4.21}$$

4.2 The dynamical constraint

So far, we have restricted the supercurl $M_{\alpha\beta}$ only by the transverse gauge condition (4.4), thereby restricting the gauge freedom (2.7). As mentioned earlier, the dynamical equations follow from imposing further linear constraints on $M_{\alpha\beta}$. We collect these into a projector K, $K^2 = K$, such that $M_{\alpha\beta}$ is subject to

$$(I-K)_{\alpha\beta}^{\alpha'\beta'}\, M_{\alpha'\beta'} \;=\; 0 \;. \tag{4.22}$$

This defines a decomposition of the superfields in $\mathcal{M}$ into

$$\mathcal{M} \;=\; \mathcal{M}_{\|} + \mathcal{M}_{\perp} \;, \qquad \text{with} \quad (I-K)\,\mathcal{M}_{\|} = 0, \quad K\mathcal{M}_{\perp} = 0 \;. \tag{4.23}$$

The role of K is twofold. First, it further restricts the field content in the superfield $M_{\alpha\beta}$ by certain algebraic relations; secondly, it implies field equations for the independent superfield components.

For example, the ordinary supersymmetric Yang-Mills equations which are equivalent to the constraint (2.9) correspond to the choice $K \to K_{\rm YM}$

$$K_{\rm YM}{}_{\alpha\beta}^{\alpha'\beta'} = \frac{1}{16}\,(\sigma^a)_{\alpha\beta}\,(\sigma_a)^{\alpha'\beta'} \;. \tag{4.24}$$

In terms of the decomposition (4.3) this corresponds to setting $B \equiv 0$:

$$F_{\alpha\beta} = 0 \;\iff\; B_{a_1\ldots a_5} = 0 \tag{4.25}$$

$$\text{i.e. for } M_{\alpha\beta}: \;\; \mathbf{10}{+}\mathbf{126} \;\mapsto\; \mathbf{10}\;.$$

At level zero, this constraint correctly selects the physical gauge potential $A_m^{[0]}$ as independent components (cf. (2.12)). At level $p=4$ the condition (4.22) then implies the Yang-Mills field equations for $A_m^{[0]}$.

Under $SO(2,1)\times SO(7)$, the field B decomposes as $\mathbf{126} \to (\mathbf{1},\mathbf{21}) + (\mathbf{3},\mathbf{35})$. The symmetrized constraint (3.11) corresponds to imposing

$$\left.\begin{array}{r} F_{\mu\nu} = 0 \\ F_{\overline{\mu}\overline{\nu}} = 0 \\ F_{\mu\overline{\nu}} + F_{\nu\overline{\mu}} = 0 \end{array}\right\} \;\iff\; \left\{\begin{array}{l} B_{+ijkl} = 0 \\ B_{-ijkl} = 0 \\ B_{8ijkl} = 0 \end{array}\right. \;, \tag{4.26}$$

$$\text{i.e. for } M_{\alpha\beta}: \;\; \underbrace{(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{7})}_{\mathbf{10}} + \underbrace{(\mathbf{1},\mathbf{21}) + (\mathbf{3},\mathbf{35})}_{\mathbf{126}} \;\mapsto\; (\mathbf{3},\mathbf{1}){+}(\mathbf{1},\mathbf{7}){+}(\mathbf{1},\mathbf{21})\;.$$

The non vanishing components of B in this case are

$$B_{ij} \;\equiv\; B_{098ij} \;=\; -\,\tfrac{1}{5!}\epsilon_{ijklmnr}B^{klmnr}\;. \tag{4.27}$$

The corresponding projector (4.22) reads

$$\begin{aligned} K_{\mu\nu}^{\mu'\nu'} &= \tfrac{1}{8}\,\delta_{\mu\nu}\,\delta^{\mu'\nu'}\;, \\ K_{\overline{\mu}\overline{\nu}}^{\overline{\mu}'\overline{\nu}'} &= \tfrac{1}{8}\,\delta_{\overline{\mu}\overline{\nu}}\,\delta^{\overline{\mu}'\overline{\nu}'}\;, \\ K_{\mu\overline{\nu}}^{\mu'\overline{\nu}'} &= \tfrac{1}{8}\,\delta_{\mu\overline{\nu}}\,\delta^{\mu'\overline{\nu}'} + \tfrac{1}{8}\,\gamma^i{}_{\mu\overline{\nu}}\,\gamma_i{}^{\mu'\overline{\nu}'} + \tfrac{1}{16}\,\gamma^{ij}{}_{\mu\overline{\nu}}\,\gamma_{ij}{}^{\mu'\overline{\nu}'}\;. \end{aligned} \tag{4.28}$$

This is a self-adjoint operator w.r.t. to the scalar product introduced in section 4.1.4 above, i.e. $K = K^{\rm ad}$.

Evidently, in addition to the YM vector potential we hence find an antisymmetric $SO(7)$ tensor field in the zero order component of the supercurl $M_{\alpha\beta}$. It is however not at all obvious which additional field content we have to expect in the higher order superfield components of $M_{\alpha\beta}$ (which with (4.25) were uniquely defined in terms of the lowest level (2.12) and thus left no space for additional fields). Nor is it clear a priori which type of field equations – if any – the weaker form of the superspace constraint implies. This is, what we are going to address in the following.

We will refer to (4.24) and (4.28) as the strong and the weak dynamical constraint, respectively.

4.2.1 Field content

To identify the physical field content among the components of the supercurl $M_{\alpha\beta}$, we collect the constraints that have been imposed on $M_{\alpha\beta}$. These are given by the transverse gauge expressed by (4.7) and the weak constraint (4.22), (4.28):

$$\begin{aligned} (S+\mathcal{R})\; M &= -\frac{\mathcal{R}+2}{\mathcal{R}}\, T\, M + \mathcal{C}\;, \\ (I-K)\, M &= 0\;. \end{aligned} \tag{4.29}$$

This obviously leaves

$$\mathcal{M}_{\|}^{+} \;\equiv\; \mathcal{M}_{\|} \cap \mathcal{M}^{+} \;=\; \ker\,(I-K) \cap \ker\,(S+\mathcal{R})\;, \tag{4.30}$$

undetermined. The independent (or physical) superfield components in $M_{\alpha\beta}$ are hence given by $\mathcal{M}_{\|}^{+}$, the space of eigenvectors of the operator KSK with eigenvalue $-p$. The remaining part of $M_{\alpha\beta}$ is consequently determined by the system (4.29) in terms of derivatives and nonlinear combinations of the physical fields. The fact that this part is in fact overdetermined by (4.29) then in turn implies the field equations as we shall discuss now.

4.2.2 Field equations

The dynamical equations arise from combining the two equations of (4.29) into

$$(S+\mathcal{R})\, M_{\|}^{[p]} + \frac{\mathcal{R}+2}{\mathcal{R}} T M_{\|}^{[p-2]} \;=\; \mathcal{C}^{[p]}\;.$$

This defines $M_{\|}^{[p]}$ in terms of the lower levels unless we project out onto vectors $\langle z|$ such that

$$\langle z|\,(S+\mathcal{R})\,M_{\|}^{[p]} = 0 = \langle z|\,(S+\mathcal{R})\,KM^{[p]}\ ,$$

in which case (4.29) implies a restriction on the image of T. We may take $\langle z| = \langle z^-|$, and obtain

$$\langle z^-|K = 0$$

Thus, the relevant vectors $\langle z^-|$ are simultaneous eigenvectors of K^{ad} with zero eigenvalue and of S^{ad} with eigenvalue 2; we denote them as $\langle z_\perp^-|$. These are eigenvectors of $\left(I-K^{\mathrm{ad}}\right)S^{\mathrm{ad}}\left(I-K^{\mathrm{ad}}\right)$ with eigenvalue 2. (As discussed above, for the weak superspace constraint we find that S and K are self-adjoint w.r.t. the scalar product induced by (4.20).) For any such eigenvector $\langle z_\perp^-|$, (4.29) yields the dynamical equation

$$\frac{p+2}{p}\left\langle z_\perp^-\right|\,TM_{\|}^{[p-2]} \;=\; \left\langle z_\perp^-\right|\,\mathcal{C}^{[p]} \tag{4.31}$$

Vice versa, if (4.31) is satisfied for all vectors of the form $\langle z_\perp^-|$, the system (4.29) has a solution for $M_{\alpha\beta}^{[p]}$ in terms of the lower levels.

Thus, the basic information about the content of the dynamical constraint (4.22) concerns the set of simultaneous eigenvectors $\langle z_\perp^-|$ and $|z_{\|}^+\rangle$, respectively. We denote the corresponding spaces by $\mathcal{M}_\perp^-$ and $\mathcal{M}_{\|}^+$, respectively. Counting of dimensions yields the identity

$$\dim\mathcal{M}_\perp - \dim\mathcal{M}^+ = \dim\mathcal{M}_\perp^- - \dim\mathcal{M}_{\|}^+\ , \tag{4.32}$$

where the numbers on the l.h.s. can simply be extracted from the representation tables of $SO(9,1)$ and $SO(7)$, respectively. For the lowest levels, these tables are collected in appendix B.

5 The physical field content.

Here, we will determine the field content which is induced by the weak dynamical constraint (4.28). For simplicity, we restrict for the rest of the paper to purely bosonic configurations, e.g. we set all fermionic fields to zero. This is just for the sake of clarity, the techniques may likewise be applied to determine the structure of the fermionic fields and field equations. In particular, supersymmetry is unbroken up to this point.

We recall, that with the strong superspace constraint (4.24), the arbitrariness in the supercurl $M_{\alpha\beta}$ is restricted to the level $p = 0$, i.e. all higher levels are determined. By analyzing the Bianchi identities for the supercurvature one verifies that in this case the following superfield relation holds [8]

$$\mathcal{R}(\mathcal{R}+1)\, M_{\alpha\beta} \;=\; \tfrac{1}{2}\,(\sigma^a)_{\alpha\beta}\,(\sigma_a{}^{bc})_{\gamma_1\gamma_2}\,\theta^{\gamma_1}\theta^{\gamma_2}\,F_{bc}\;, \tag{5.1}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_b + [A_a, A_b]_-\;,$$

is now the curvature of the superfield A_a. Together with (4.6), one hence obtains recurrence relations which completely determine $M_{\alpha\beta}$ in terms of its lowest components – the physical fields X_a.

With the weak dynamical constraint (4.28) the situation becomes essentially more complicated. In particular, more fields have to be introduced to eventually obtain closed recurrence relations which would replace (5.1). In this section we take the first step and analyze the physical field content in the superfield $M_{\alpha\beta}$ as implied by the weak dynamical constraint.

To keep things tractable, we will most of the time restrict the analysis to those fields which transform as singlets under the $SO(2,1)$ symmetry underlying (3.10). As we have shown in section 3, this is consistent truncation of the system to seven dimensions. In this spin 0 sector we can determine the complete field content. In addition to the Yang-Mills fields, we find certain additional vector and antisymmetric tensor fields, collected in (5.26), below. Despite the technical restriction to $SO(2,1)$ singlets, the method described in the following allows straight-forward although more tedious generalization to the higher spin fields.

According to the general discussion above, the independent components in the supercurl $M_{\alpha\beta}$ are given by the space $\mathcal{M}^+_\parallel$, i.e. by the intersection of the kernels of $(S+\mathcal{R})$ and $(I-K)$. We start from (4.32)

$$\dim \mathcal{M}^+_\parallel \;=\; \dim \mathcal{M}^+ - \dim \mathcal{M}_\perp + \dim \mathcal{M}^-_\perp\;, \tag{5.2}$$

and will in the next two sections determine the r.h.s. of this equation when restricted to $SO(2,1)$ singlets. As this part is necessarily somewhat technical, it may be skipped on a first reading; the results are summarized and discussed in section 5.3 below.

5.1 The space of $SO(2,1)$ singlets

The supercurl $M_{\alpha\beta}$ may be decomposed according to the spin of $SO(2,1)$. Recalling the notation of section 3.3, and equation (3.14) in particular, the generators of $SO(2,1)$ act on the supercurl $M_{\alpha\beta}$ as

$$\delta_k M \;=\; J_k M + M J_k^T + (J_k)_\rho^{\ \sigma}\, \theta^\rho \partial_\sigma M \; . \tag{5.3}$$

We label the total $SO(2,1)$ spin by ℓ and its z-component (i.e. the eigenvalue of δ_0 which is raised resp. lowered by $\delta_\pm$) by ℓ_0. According to the action of δ_0, the value of ℓ_0 is given by the difference of barred and unbarred indices in a superfield $M_{\alpha\beta,\gamma_1\ldots\gamma_p}$. The space of $SO(2,1)$ singlets $\mathcal{M}_{\ell=0}$ is obtained from the space with vanishing z-component, $\mathcal{M}_{\ell_0=0}$, by dividing out the action of $\delta_\pm$. Equivalently, one has the relation

$$\dim \mathcal{M}_{\ell=0} \;=\; \dim \mathcal{M}_{\ell_0=0} - \dim \mathcal{M}_{\ell_0=1} \; . \tag{5.4}$$

The latter characterization will moreover be helpful to obtain the remaining $SO(7)$ representation structure of $\mathcal{M}_{\ell=0}$.

Specifically, $\mathcal{M}^{[p]}_{\ell_0=0}$ is spanned by vectors

$$\left\{ \begin{array}{l} M_{\mu\nu,\mu_1\ldots\mu_{k-1},\bar\rho_1\ldots\bar\rho_{k+1}} \\ M_{\mu\bar\nu,\mu_1\ldots\mu_k,\bar\rho_1\ldots\bar\rho_k} \\ M_{\overline{\mu\nu},\mu_1\ldots\mu_{k+1},\bar\rho_1\ldots\bar\rho_{k-1}} \end{array} \right. \qquad \text{with} \quad 2k=p \; , \tag{5.5}$$

and the space of singlets $\mathcal{M}^{[p]}_{\ell=0}$ is obtained from (5.5) by modding out the action of the generator δ_+:

$$\begin{aligned} M_{\mu\nu,\mu_1\ldots\mu_{k-2}[\mu_{k-1},\bar\rho_1\ldots\bar\rho_{k+1}]} &= 0 \\ k\, M_{\mu\bar\nu,\mu_1\ldots\mu_{k-1}[\mu_k,\bar\rho_1\ldots\bar\rho_k]} &= -M_{\mu\nu,\mu_1\ldots\mu_{k-1},\bar\rho_1\ldots\bar\rho_{k+1}} \\ k\, M_{\nu\bar\mu,\mu_1\ldots\mu_{k-1}[\mu_k,\bar\rho_1\ldots\bar\rho_k]} &= -M_{\mu\nu,\mu_1\ldots\mu_{k-1},\bar\rho_1\ldots\bar\rho_{k+1}} \\ (k+1)\, M_{\overline{\mu\nu},\mu_1\ldots\mu_k[\mu_{k+1},\bar\rho_1\ldots\bar\rho_{k-1}]} &= -M_{\mu\bar\nu,\mu_1\ldots\mu_k,\bar\rho_1\ldots\bar\rho_k} - M_{\nu\bar\mu,\mu_1\ldots\mu_k,\bar\rho_1\ldots\bar\rho_k} \; . \end{aligned} \tag{5.6}$$

Similar relations are obtained from the action of δ_- on the vectors (5.5), these may however already be deduced from (5.6) and are not independent. In particular, one derives the symmetry relations

$$\begin{aligned} M_{\mu\nu,\mu_1\ldots\mu_{k-1},\bar\rho_1\ldots\bar\rho_{k+1}} &= M_{\overline{\mu\nu},\rho_1\ldots\rho_{k+1},\bar\mu_1\ldots\bar\mu_{k-1}} \\ M_{\mu\bar\nu,\mu_1\ldots\mu_k,\bar\rho_1\ldots\bar\rho_k} &= M_{\nu\bar\mu,\rho_1\ldots\rho_k,\bar\mu_1\ldots\bar\mu_k} \end{aligned} \tag{5.7}$$

The relations (5.6) show that in fact, the $SO(2,1)$ singlets are parametrized by their part $M_{\mu\overline{\nu}}$ (together with a symmetrization condition), which determines $M_{\mu\nu}$ and $M_{\overline{\mu}\overline{\nu}}$. For the analysis of the field content, it turns out to be helpful to introduce (yet) another decomposition of $\mathcal{M}$:

$$\mathcal{M} \;=\; \mathcal{N} + \widetilde{\mathcal{N}} \qquad \text{with} \quad \begin{cases} \mathcal{N} = \{M_{\mu\overline{\nu}} + M_{\nu\overline{\mu}}\} \\ \widetilde{\mathcal{N}} = \{M_{\mu\overline{\nu}} - M_{\nu\overline{\mu}}\} \end{cases} \tag{5.8}$$

This decomposition is orthogonal w.r.t. the scalar product (4.20). The field content of the superfield $M_{\alpha\beta}$ has been essentially encoded in the operators S and K from (4.8) and (4.28), respectively. We denote by $S_{\mathcal{N}}$ and $K_{\mathcal{N}}$, respectively, their projections onto $\mathcal{N}$ according to the decomposition (5.8)

$$S_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N} \;, \qquad K_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N} \;.$$

In analogy to (4.12) and (4.23), we define the decompositions of $\mathcal{N}$ into eigenspaces of $S_{\mathcal{N}}$ and $K_{\mathcal{N}}$ by

$$\mathcal{N} \;=\; \mathcal{N}^{+} + \mathcal{N}^{-} \;, \qquad \text{with} \quad (S_{\mathcal{N}} - 2)\,\mathcal{N}^{-} \;=\; 0 \;, \tag{5.9}$$

and

$$\mathcal{N} \;=\; \mathcal{N}_{\perp} + \mathcal{N}_{\parallel} \;, \qquad \text{with} \quad K_{\mathcal{N}}\mathcal{N}_{\perp} \;=\; 0 \;. \tag{5.10}$$

It is then easy to observe that (5.8) is compatible with (and somewhat tailored for) the weak dynamical constraint (4.28) in the sense that K and $K_{\mathcal{N}}$ commute and have coinciding eigenvalues with

$$\mathcal{M}_{\perp} = \mathcal{N}_{\perp} \;, \qquad \mathcal{N}_{\parallel} \subset \mathcal{M}_{\parallel} \;. \tag{5.11}$$

The operator $S_{\mathcal{N}}$ however does not commute with its ancestor S and correspondingly has eigenvalues which do not necessarily coincide with those of S. However, since (5.8) is an orthogonal decomposition, it follows that the eigenvalues of $S_{\mathcal{N}}$ lie in the interval $[-p, 2]$. Moreover, eigenvectors of $S_{\mathcal{N}}$ with eigenvalues $-p$ and 2, respectively, are necessarily also eigenvectors of S.

5.2 The relevant subspaces

Equations (5.8), (5.9) and (5.11) allow to rewrite (5.2) as

$$\begin{aligned} \dim \mathcal{M}^+_{\parallel} &= \dim \mathcal{M}^+ - \dim \mathcal{N}_{\perp} + \dim \mathcal{N}^-_{\perp} \\ &= \dim \mathcal{M}^+ - \dim \mathcal{N}^+ + \mathcal{N}^+_{\parallel} \ . \end{aligned} \tag{5.12}$$

In the following, we are going to determine the three spaces on the r.h.s. restricted to the sector of $SO(2,1)$ singlets.

5.2.1 The space $\mathcal{M}^+_{\ell=0}$

Here, we determine the decomposition (4.12) on the space of $SO(2,1)$ singlets

$$\mathcal{M}^{[p]}_{\ell=0} = \mathcal{M}^{+\,[p]}_{\ell=0} + \mathcal{M}^{-\,[p]}_{\ell=0} \ . \tag{5.13}$$

The corresponding decomposition of $\mathcal{M}^{[p]}_{\ell_0}$ with fixed ℓ_0 may be obtained from (4.13) by analyzing the decomposition of the Young tableaux under $\mathbf{16} \mapsto \mathbf{8}+\mathbf{8}$. Together with (5.4), the dimension of the space $\mathcal{M}^+_{\ell=0}$ is then given by (depicted as an illustration for $p=8$, but likewise valid for all even values of $p \geq 2$)

$$\begin{aligned} \mathcal{M}^{+\,[p]}_{\ell=0} &= \underbrace{\left(\ydiagram{1,1,1,1,1} \times \ydiagram{1,1,1,1,1} + \ydiagram{1,1,1,1,1} \times \ydiagram{2,1,1,1} + \ydiagram{2,1,1,1} \times \ydiagram{1,1,1,1,1} \right)}_{\mathcal{M}^{+\,[p]}_{\ell_0=0}} - \underbrace{\left(\ydiagram{1,1,1,1,1,1} \times \ydiagram{1,1,1,1} + \ydiagram{1,1,1,1,1,1} \times \ydiagram{2,1,1} + \ydiagram{2,1,1,1,1} \times \ydiagram{1,1,1,1} \right)}_{\mathcal{M}^{+\,[p]}_{\ell_0=1}} \\ &= \ydiagram{2,2,2,2,1,1} + \ydiagram{3,2,2,2,1} \end{aligned} \tag{5.14}$$

where each box now represents a $\mathbf{8}$ of $SO(7)$. The second equality in (5.14) is obtained from standard Young tableaux manipulations. In particular, this gives the dimension

$$\dim \mathcal{M}^{+\,[p]}_{\ell=0} = \binom{8}{k+1} \binom{9}{k} \frac{17k+7}{(k+1)(k+2)} \ . \tag{5.15}$$

5.2.2 The space $\mathcal{N}^+_{\ell=0}$

The space $\mathcal{N}$ is organized such that the $SO(2,1)$ singlets in $\mathcal{N}$ are parametrized by their part $M_{\mu\nu}$ together with the condition

$$M_{\mu\nu,\mu_1\ldots\mu_{k-2}[\mu_{k-1},\overline{\rho}_1\ldots\overline{\rho}_{k+1}]} \;=\; 0\;. \tag{5.16}$$

By means of the relations (5.6), this determines $M_{\mu\overline{\nu}}$ and $M_{\overline{\mu\nu}}$. In particular, this allows to describe the action of the operator $S_{\mathcal{N}}$ entirely in terms of its action onto $M_{\mu\nu}$. Combining (4.8) and (5.6) one obtains after some calculation

$$\begin{aligned}
(S_{\mathcal{N}}\,M)_{\mu\nu} \;=\;& 2(k-1)\,M_{\mu_1(\mu,\nu)\mu_2\ldots\mu_{k-1},\overline{\rho}_1\ldots\overline{\rho}_{k+1}}\,\theta^{\mu_1}\ldots\theta^{\mu_{k-1}}\theta^{\overline{\rho}_1}\ldots\theta^{\overline{\rho}_{k+1}} \\
&+(k^2-1)\,M_{\overline{\rho}_1(\mu,\nu)\mu_2\ldots\mu_{k-1},\mu_1\overline{\rho}_2\ldots\overline{\rho}_{k+1}}\,\theta^{\mu_1}\ldots\theta^{\mu_{k-1}}\theta^{\overline{\rho}_1}\ldots\theta^{\overline{\rho}_{k+1}} \\
&-(k+1)\,M_{\overline{\rho}_1(\mu|,\mu_1\ldots\mu_{k-1},|\nu)\overline{\rho}_2\ldots\overline{\rho}_{k+1}}\,\theta^{\mu_1}\ldots\theta^{\mu_{k-1}}\theta^{\overline{\rho}_1}\ldots\theta^{\overline{\rho}_{k+1}}\;.
\end{aligned} \tag{5.17}$$

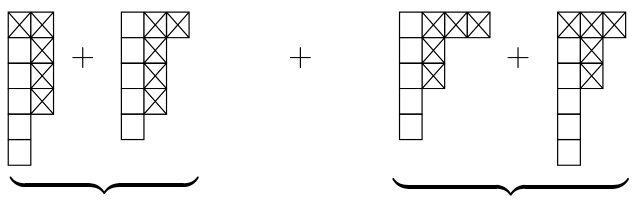
Since $S_{\mathcal{N}}$ acts by merely permuting indices, it already follows from the form of the condition (5.16) that it can have no more than four distinct eigenspaces which are given by certain Young tableaux symmetrizations in the $(2k+2)$ indices. More precisely, one finds – similarly to (4.11) – the equation

$$(S_{\mathcal{N}}-2)\,(S_{\mathcal{N}}-(2-k))\,(2S_{\mathcal{N}}-(1-3k)) \;=\; 0\;, \tag{5.18}$$

which determines the eigenvalues of $S_{\mathcal{N}}$. The corresponding eigenspaces are given by the following Young tableaux (again depicted for $p=8$, but likewise valid for all even values of $p\geq 2$)

eigenvalues of $S_{\mathcal{N}}$: $\quad 2-k \quad \frac{1}{2}(1-3k) \quad 2 \quad 2$ (5.19)

$\mathcal{N} \;=$



$\mathcal{N}^+$ $\qquad$ $\mathcal{N}^-$

Blank and crossed boxes here represent barred and unbarred indices, respectively, referring both to the **8** of $SO(7)$.

The space $\mathcal{N}^+$ is the sum of eigenspaces of $S_\mathcal{N}$ with eigenvalues other than 2, i.e. according to (5.18) with eigenvalues $(2-k)$ and $\frac{1}{2}(1-3k)$, respectively. Comparing the Young tableaux from (5.19) with (5.14) we find that

$$\dim \mathcal{N}^+_{\ell=0} \;=\; \dim \mathcal{M}^+_{\ell=0}\;. \tag{5.20}$$

This is a key equation in our analysis of the field content. According to (5.12) it reduces the problem of determining the field content in the superfield $M_{\alpha\beta}$ to the space $\mathcal{N}^+_\parallel$ which has very simple properties and may explicitly be given as we shall show now.

5.2.3 The space $\mathcal{N}^+_{\parallel\,\ell=0}$

The space $\mathcal{N}^+_\parallel$ is defined to be the intersection of $\mathcal{N}^+$ and $\mathcal{N}_\parallel$. For $p=2$, it follows from (5.17) and (5.19) that $\mathcal{N}=\mathcal{N}^+$ and hence $\mathcal{N}^+_\parallel=\mathcal{N}_\parallel$ is nonempty. Explicitly, it is

$$\mathcal{N}^{[2]}_\parallel \;=\; \Big\{ M_{\mu\nu,\bar\rho_1\bar\rho_2} \;=\; \delta_{\mu\nu}\,\gamma^i_{\bar\rho_1\bar\rho_2}\,Z_i+\delta_{\mu\nu}\,\gamma^{ij}_{\bar\rho_1\bar\rho_2}\,C_{ij}\Big\} \;\sim\; \mathbf{7}+\mathbf{21}\;. \tag{5.21}$$

In contrast to the structure with the strong dynamical constraint (4.24), the weak constraint (4.28) hence leaves some components of the level $p=2$ undetermined. This means that additional fields are arising and will later on appear to be coupled to the original Yang-Mills fields.

We will now show that at the higher levels $p\geq 4$, the space $\mathcal{N}^+_\parallel$ is empty and hence no further physical fields arise; the field content of the model is thus completely contained in the lowest two levels of the supercurl. According to (4.28), the space $\mathcal{N}_\parallel$ is given by the restriction of superfields $M_{\mu\nu}$ onto their trace part in the indices $(\mu\nu)$. It follows from the form of the conditions (5.6) that the operator $(K_\mathcal{N}\,S_\mathcal{N}\,K_\mathcal{N})$ has only one nonvanishing eigenvalue which is determined from (5.17)

$$(K_\mathcal{N}\,S_\mathcal{N}\,K_\mathcal{N}) \;=\; \tfrac{1}{4}\,(k-2)\,K_\mathcal{N}\;. \tag{5.22}$$

Since $K_\mathcal{N}$ is an orthogonal projector, comparing this relation with the eigenvalues of $S_\mathcal{N}$ on $\mathcal{N}^+$ shows, that $\mathcal{N}^+_\parallel$ can be nonempty only if

$$\tfrac{1}{2}\,(1-3k) \;\leq\; \tfrac{1}{4}\,(k-2) \;\leq\; (2-k) \qquad\Longrightarrow\qquad p=2k\;\leq\;4\;. \tag{5.23}$$

For $k=2$ (i.e. $p=4$) one may furthermore show by a similar but slightly more complicated analysis of the operator $(K_\mathcal{N}\,S^2_\mathcal{N}\,K_\mathcal{N})$, that also $\mathcal{N}^{+\,[4]}_\parallel$ is empty. We leave details to the reader.

5.3 The field content

Summarizing these results, we have proven that the space $\mathcal{M}^{+}_{\parallel\,\ell=0}$ is empty for all levels $p \geq 4$. According to the general discussion of the section 4, this shows that physical fields arise only at the lowest two levels of the supercurl $M_{\alpha\beta}$ whereas all higher order levels are determined by the system (4.29) as functions of the physical fields.

Let us collect the physical field content. At $p=0$, clearly, $SM^{[0]} = 0$, $\mathcal{R}M^{[0]} = 0$. Thus $M^{[0]} = M^{[0]+}$ and the level zero is hence completely arbitrary. The physical fields are projected out by (4.22). We find the supercurl $M_{\alpha\beta}$ to contain the independent components X_i and B_{ij} of which the former is part of the original Yang-Mills vector potential whereas the latter represents an additional antisymmetric tensor field

$$M^{[0]}_{\mu\overline{\nu}} \;=\; \gamma^{i}{}_{\mu\overline{\nu}}\, X_i + \gamma^{ij}{}_{\mu\overline{\nu}}\, B_{ij} \;\sim\; \mathbf{7} + \mathbf{21}\;. \tag{5.24}$$

We recall, that B_{ij} appears additionally in this case as compared to the strong superfield constraint (4.24). This has of course already been observed in (4.27).

At level $p=2$ we have determined the size of the space of physical fields by computing the space $\mathcal{N}_{\parallel}$ and making use of (5.12) and (5.20). We find additional fields which we denote by Z_i and C_{ij}. Explicitly, they arise as

$$\begin{aligned}
M^{[2]}_{\mu\nu} &= 8\,\delta_{\mu\nu}\,\gamma^{i}_{\overline{\rho}_1\overline{\rho}_2}\,\theta^{\overline{\rho}_1}\theta^{\overline{\rho}_2}\;Z_i + 8\,\delta_{\mu\nu}\,\gamma^{ij}_{\overline{\rho}_1\overline{\rho}_2}\,\theta^{\overline{\rho}_1}\theta^{\overline{\rho}_2}\;C_{ij} + \ldots \qquad (5.25)\\
M^{[2]}_{\mu\overline{\nu}} &= -\Big(4\,\delta_{\mu\overline{\nu}}\,\gamma^{i}_{\rho_1\overline{\rho}_2} + 2\,\delta_{\rho_1\overline{\rho}_2}\,\gamma^{i}_{\mu\overline{\nu}} + \gamma^{imn}_{\rho_1\overline{\rho}_2}\,\gamma^{mn}_{\mu\overline{\nu}}\Big)\;\theta^{\rho_1}\theta^{\overline{\rho}_2}\;Z_i\, - \\
&\quad -\Big(4\,\delta_{\mu\overline{\nu}}\,\gamma^{ij}_{\rho_1\overline{\rho}_2} + 2\,\delta_{\rho_1\overline{\rho}_2}\,\gamma^{ij}_{\mu\overline{\nu}} + 2\,\gamma^{ijm}_{\rho_1\overline{\rho}_2}\,\gamma^{m}_{\mu\overline{\nu}} + \gamma^{ijmn}_{\rho_1\overline{\rho}_2}\,\gamma^{mn}_{\mu\overline{\nu}}\Big)\;\theta^{\rho_1}\theta^{\overline{\rho}_2}\;C_{ij}\, + \\
&\quad + \ldots \\
M^{[2]}_{\overline{\mu}\overline{\nu}} &= 8\,\delta_{\overline{\mu}\overline{\nu}}\,\gamma^{i}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;Z_i + 8\,\delta_{\overline{\mu}\overline{\nu}}\,\gamma^{ij}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;C_{ij} + \ldots
\end{aligned}$$

where the dots "…" denote the terms that are determined by derivatives and bilinear combinations of the fields from the lowest level. These contributions are given explicitly in equation (6.4), below.

Summarizing, we have shown that in the sector of $SO(2,1)$ singlets, the supercurl $M_{\alpha\beta}$ in transverse gauge and with the weak superspace constraint (4.28) imposed, is determined in all orders by the set of physical fields

$$X_i\,,\; B_{ij}\,,\; Z_i\,,\; C_{ij}\;, \tag{5.26}$$

which enter as components at the levels $p=0$ and $p=2$ of the superfield expansion of $M_{\alpha\beta}$. In the following, we will study what kind of dynamical relations we may in addition extract for these fields.

6 Field equations.

In this section we will determine the field equations implied by the weak dynamical constraint (4.28) onto the fields (5.26). To this end, we explicitly analyze the lowest superfield levels of $M_{\alpha\beta}$. Following the general discussion of section 4, the dynamical equations arise from projecting the image of the operator T according to (4.31) onto $\mathcal{M}_{\perp}^{-}$.

In the foregoing analysis of the field content we were able to consistently truncate the problem to the subspace of $SO(2,1)$ singlets, since the operators S and K encoding the field content via (4.29) transform as singlets under the $SO(2,1)\times SO(7)$. Once we are taking the full system (4.29) into account, this truncation fails due to the nontrivial transformation behavior of the level raising operator T. Since T depends linearly on the spatial derivatives

$$T_{\alpha\beta}^{\alpha'\beta'} \;\equiv\; \sum_a (T^a)_{\alpha\beta}^{\alpha'\beta'}\,\partial_a \;, \tag{6.1}$$

it apparently contains parts transforming as $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{7})$, respectively, under the $SO(2,1)\times SO(7)$. In particular, the T^i preserve the $SO(2,1)$ spin whereas $(T^{\pm},T^8)$ transforms as a vector under $SO(2,1)$. Correspondingly, T^+ and T^- raise and lower, respectively, the $z-$component of the $SO(2,1)$ spin.

We may hence restrict the analysis of the field content and dynamical equations described above to the corresponding chains of subspaces; however, starting from $SO(2,1)$ singlets at the lowest level will then certainly involve higher spin fields in the higher superfield components. For simplicity, we will hence in the following treat the system upon dimensional reduction to seven dimensions. Dropping the coordinate dependence of all fields on x^8, $x^{\pm}$ implies that also the operator T transforms as an $SO(2,1)$ singlet. In this case we may consistently truncate the system (4.29) neglecting all fields with higher $SO(2,1)$ spins. In section 7 below, we comment on the extension of the analysis to higher spin fields.

6.1 Level $p = 2$

With the strong constraint (4.24) imposed, the level $p=2$ is uniquely determined by the physical fields of the lower levels by the lowest order component of equation (5.1)

$$M^{[2]}_{\alpha\beta} = (\sigma^a)_{\alpha\beta}\,(\sigma_a{}^{bc})_{\gamma_1\gamma_2}\,\theta^{\gamma_1}\theta^{\gamma_2}\,Y_{bc}\,. \tag{6.2}$$

where

$$Y_{ab} = \partial_a X_b - \partial_b X_a + [X_a,\, X_b]_-\,. \tag{6.3}$$

In particular, neither are there new fields arising in the level $p=2$ components of $M_{\alpha\beta}$, nor do we obtain any restrictions on the fields of the lower levels.

Let us turn to the weak dynamical constraint. As we have shown above, at the level $p=2$ of the superfield $M_{\alpha\beta}$ we find new fields arising. Here, we complement the formulae (5.25) by explicitly giving the terms which are determined by derivatives and bilinear combinations of the lowest order fields B_{ij} and X_i:

$$\begin{aligned}
M^{[2]}_{\mu\nu} \;=\;& 8\,\delta_{\mu\nu}\,\gamma^{i}_{\bar\rho_1\bar\rho_2}\,\theta^{\bar\rho_1}\theta^{\bar\rho_2}\;Z_i + 8\,\delta_{\mu\nu}\,\gamma^{ij}_{\bar\rho_1\bar\rho_2}\,\theta^{\bar\rho_1}\theta^{\bar\rho_2}\;C_{ij} + \\
&+ \delta_{\mu\nu}\,\gamma^{ij}_{\bar\rho_1\bar\rho_2}\,\theta^{\bar\rho_1}\theta^{\bar\rho_2}\;Y_{ij} - \tfrac{2}{5}\,\delta_{\mu\nu}\,\gamma^{ij}_{\bar\rho_1\bar\rho_2}\,\theta^{\bar\rho_1}\theta^{\bar\rho_2}\;[B_{ik}, B_{jk}] + \\
&+ \tfrac{4}{3}\,\delta_{\mu\nu}\,\gamma^{i}_{\bar\rho_1\bar\rho_2}\,\theta^{\bar\rho_1}\theta^{\bar\rho_2}\;D^j B_{ij}
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
M^{[2]}_{\mu\bar\nu} \;=\;& -\Big(4\,\delta_{\mu\bar\nu}\,\gamma^{i}_{\rho_1\bar\rho_2} + 2\,\delta_{\rho_1\bar\rho_2}\,\gamma^{i}_{\mu\bar\nu} + \gamma^{imn}_{\rho_1\bar\rho_2}\,\gamma^{mn}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;Z_i - \\
&- \Big(4\,\delta_{\mu\bar\nu}\,\gamma^{ij}_{\rho_1\bar\rho_2} + 2\,\delta_{\rho_1\bar\rho_2}\,\gamma^{ij}_{\mu\bar\nu} + 2\,\gamma^{ijm}_{\rho_1\bar\rho_2}\,\gamma^{m}_{\mu\bar\nu} + \gamma^{ijmn}_{\rho_1\bar\rho_2}\,\gamma^{mn}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;C_{ij} - \\
&- \tfrac{1}{2}\,\Big(\delta_{\mu\bar\nu}\,\gamma^{ij}_{\rho_1\bar\rho_2} + \gamma^{ijm}_{\rho_1\bar\rho_2}\,\gamma^{m}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;Y_{ij} + \\
&+ \tfrac{1}{5}\,\Big(\delta_{\mu\bar\nu}\,\gamma^{ij}_{\rho_1\bar\rho_2} - 2\,\delta_{\rho_1\bar\rho_2}\,\gamma^{ij}_{\mu\bar\nu} + \gamma^{ijm}_{\rho_1\bar\rho_2}\,\gamma^{m}_{\mu\bar\nu} + 2\gamma^{ijmn}_{\rho_1\bar\rho_2}\,\gamma^{mn}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;[B_{ik}, B_{jk}] + \\
&+ \Big(\gamma^{ijk}_{\rho_1\bar\rho_2}\,\gamma^{l}_{\mu\bar\nu} + \gamma^{ijkm}_{\rho_1\bar\rho_2}\,\gamma^{lm}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;[B_{ij}, B_{kl}] - \\
&- \tfrac{1}{3}\,\Big(2\,\delta_{\mu\bar\nu}\,\gamma^{i}_{\rho_1\bar\rho_2} - 2\,\delta_{\rho_1\bar\rho_2}\,\gamma^{i}_{\mu\bar\nu} - \gamma^{imn}_{\rho_1\bar\rho_2}\,\gamma^{mn}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;D^j B_{ij} + \\
&+ \tfrac{1}{2}\,\Big(\gamma^{ijm}_{\rho_1\bar\rho_2}\,\gamma^{km}_{\mu\bar\nu} + 7\,\gamma^{ijkm}_{\rho_1\bar\rho_2}\,\gamma^{m}_{\mu\bar\nu} + 6\,\gamma^{kim}_{\rho_1\bar\rho_2}\,\gamma^{jm}_{\mu\bar\nu}\Big)\;\theta^{\rho_1}\theta^{\bar\rho_2}\;D_k\,B_{ij}
\end{aligned}$$

$$
\begin{aligned}
M^{[2]}_{\overline{\mu\nu}} &= 8\,\delta_{\overline{\mu\nu}}\,\gamma^{i}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;Z_i + 8\,\delta_{\overline{\mu\nu}}\,\gamma^{ij}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;C_{ij} + \\
&\quad + \delta_{\overline{\mu\nu}}\,\gamma^{ij}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;Y_{ij} - \tfrac{2}{5}\,\delta_{\overline{\mu\nu}}\,\gamma^{ij}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;[B_{ik}, B_{jk}] + \\
&\quad + \tfrac{4}{3}\,\delta_{\overline{\mu\nu}}\,\gamma^{i}_{\rho_1\rho_2}\,\theta^{\rho_1}\theta^{\rho_2}\;D^j B_{ij}
\end{aligned}
$$

No field equations arise at this level, i.e. the projection (4.31) turns out to be satisfied without imposing any restrictions on the level zero fields.

6.2 Level $p = 4$

On this level, we expect the dynamical equations for the lowest level fields. In particular, the strong dynamical constraint is known to imply the bosonic Yang-Mills equations of motion on this level. The weaker constraint correspondingly should give rise to some analogue for the fields (5.26).

Let us first recapitulate the case of the strong constraint. The group decomposition of superfields on this level is given in table (B.4). It shows already that $\mathcal{M}^{-}_{\perp}$ is nonempty but contains e.g. the vector representation **10** of $SO(9,1)$ arising on this level. According to (4.31) the dynamical equation is given by the scalar product

$$
\Big\langle \mathcal{M}^{-\,[4]}_{\perp\,\mathbf{10}} \Big|\; \Big(TM^{[2]}_{\parallel} - \tfrac{2}{3}\mathcal{C}^{[4]} \Big)\;, \qquad \text{with} \quad TM^{[2]}_{\parallel} \sim \partial^b\, Y_{ab} \tag{6.5}
$$

Since this is the nondegenerate scalar product on a space of multiplicity one, it suffices to show that $TM^{[2]}_{\parallel} \neq 0$ (with $M^{[2]}_{\parallel}$ given by (6.2)) to indeed obtain the bosonic part of the Yang-Mills field equations

$$
\mathcal{D}^b\, Y_{ab} \;\equiv\; \partial^b\, Y_{ab} + \left[A^b\,, Y_{ab}\right] \;=\; 0\;. \tag{6.6}
$$

One might expect to find further relations in the **120** and **126**, respectively, from which according to (B.4) the space $\mathcal{M}^{-}_{\perp}$ also has nonvanishing contributions. However, the first one contains precisely the Bianchi identities of Y_{ab} which are automatically satisfied, whereas there is no nontrivial image of T into the **126** as one may easily verify. Thus, in agreement with [8], there arise no further restrictions than the Yang-Mills equations of motion (6.6), here.

We now turn to the weak dynamical constraint. Having shown that $\mathcal{M}^{+}_{\parallel} = 0$ on all higher levels $p \geq 4$, we may invert the relation (4.32) to find

$$
\dim \mathcal{M}^{-}_{\perp} \;=\; \dim \mathcal{M}_{\perp} - \dim \mathcal{M}^{+}\;, \tag{6.7}
$$

where the numbers on the r.h.s. may be extracted from the representation tables collected in appendix B.

We first consider the $SO(7)$ singlet $\mathbf{1}$ which appears in a particularly simple way in (B.5), namely with multiplicity one. Moreover, the table shows that

$$\text{mult}\, \mathcal{M}^{-\,[4]}_{\perp\,\mathbf{1}} \;=\; 1 \;, \tag{6.8}$$

i.e. according to (4.31), a dynamical equation arises from the scalar product

$$\Big\langle \mathcal{M}^{-\,[4]}_{\perp\,\mathbf{1}} \Big|\; \big(TM^{[2]} - \tfrac{2}{3}\mathcal{C}^{[4]}\big) \;, \qquad \text{with } \; M^{[2]} \; \text{given by (6.4)} \;. \tag{6.9}$$

Similarly to (6.5), this scalar product is particularly simple to compute because it lives on a space $\mathcal{M}^{[4]}_{\mathbf{1}}$ of multiplicity one. With the explicit expression from (6.4) we arrive at the first field equation for the enlarged system

$$\mathcal{D}^i Z_i \;=\; -\left[B^{ij}, Y_{ij} + 2C_{ij}\right] \;. \tag{6.10}$$

Note, that this equation has no analogue in the original Yang-Mills system since in that system there is no combination of fields and derivatives transforming as singlet of $SO(9,1)$ at this order.

Let us continue with the vector part $\mathbf{7}$ which should contain the analogue of the Yang-Mills equations of motion. For illustration, we will describe this sector in some detail. According to the general proceeding outlined above, we first determine the subspace $\mathcal{M}^{-}_{\perp}$ which by projection gives rise to the dynamical equations of the system. It follows from (B.5) and (6.7) that $\mathcal{M}^{-\,[4]}_{\perp\,\mathbf{7}}$ is nonempty with multiplicity one. To determine this space explicitly, it suffices to diagonalize the operator S from (4.8) on the space $\mathcal{M}_{\perp\,\mathbf{7}}^{[4]}$. A basis of the latter is e.g. given by

$$\begin{aligned}
(w_1)^i_{\mu_1\bar\rho_1\,,\,\mu_2\mu_3\bar\rho_2\bar\rho_3} &= \gamma^{mnk}_{\mu_1\bar\rho_1}\left(\gamma^{mn}_{\mu_2\mu_3}\,\gamma^{ki}_{\bar\rho_2\bar\rho_3} - \gamma^{ki}_{\mu_2\mu_3}\,\gamma^{mn}_{\bar\rho_2\bar\rho_3}\right) \;, \\
(w_2)^i_{\mu_1\bar\rho_1\,,\,\mu_2\mu_3\bar\rho_2\bar\rho_3} &= \gamma^{imnk}_{\mu_1\bar\rho_1}\left(\gamma^{mn}_{\mu_2\mu_3}\,\gamma^{k}_{\bar\rho_2\bar\rho_3} - \gamma^{k}_{\mu_2\mu_3}\,\gamma^{mn}_{\bar\rho_2\bar\rho_3}\right) \;, \\
(w_3)^i_{\mu_1\bar\rho_1\,,\,\mu_2\mu_3\bar\rho_2\bar\rho_3} &= \gamma^{imn}_{\mu_1\bar\rho_1}\,\gamma^{m}_{\mu_2\mu_3}\,\gamma^{n}_{\bar\rho_2\bar\rho_3} \;, \\
(w_4)^i_{\mu_1\bar\rho_1\,,\,\mu_2\mu_3\bar\rho_2\bar\rho_3} &= \gamma^{imn}_{\mu_1\bar\rho_1}\,\gamma^{mk}_{\mu_2\mu_3}\,\gamma^{nk}_{\bar\rho_2\bar\rho_3} \;,
\end{aligned} \tag{6.11}$$

where the other components of these vectors are obtained from the conditions (5.6), discussed above. Computing the action of S on this basis (6.11) one finds that $\mathcal{M}^{-\,[4]}_{\perp\,\mathbf{7}}$ is spanned by

$$\mathcal{M}^{-\,[4]}_{\perp\,\mathbf{7}} \;=\; \left\{w^i_1 - w^i_2 + 4w^i_3\right\} \;. \tag{6.12}$$

The dynamical equations are finally obtained according to (4.31) by projecting the image of $M_{\|}^{[2]}$ – the latter being entirely given by (6.4) – under T onto the constraint vector (6.12)

$$\left\langle \mathcal{M}^{-[4]}_{\perp \mathbf{7}} \;\middle|\; \left(T M_{\|}^{[2]} - \tfrac{2}{3}\mathcal{C}^{[4]} \right)\right. .$$

This computation has been performed on the computer and yields the following result

$$\begin{aligned} 15\,\mathcal{D}^m\left(Y_{mi} + C_{mi}\right) &= -8\,[B_{im}, \mathcal{D}_n B^{mn}] - 5\,[B_{im}, Z^m] - \\ &\quad - 60\,[B^{mn}, \mathcal{D}_m B_{ni}] - 24\,\,[B^{mn}, \mathcal{D}_i B_{mn}]\;. \end{aligned} \tag{6.13}$$

This gives the extension of the Yang-Mills equations for the enlarged system associated to the weak dynamical constraint and reduces to the former in absence of the additional fields B_{ij}, Z_i, C_{ij}.

Similarly, one may continue with all the other $SO(7)$ subrepresentations contained in $M_{\alpha\beta}$ at this level. In the lowest dimensional parts, we find in addition another (first-order) field equation in the **35** of $SO(7)$:

$$\begin{aligned} \mathcal{D}_{[i}\,C_{jk]} - \tfrac{1}{4}\left[Z_{[i}\,, B_{jk]}\right] &= \epsilon_{ijkmnpq}\left[B^{mn}\,,\, \kappa_1 Y^{pq} + \kappa_2 C^{pq} + \kappa_3\,[B^{pr}, B^q{}_r]\right] \\ &\quad + \kappa_4\left[B_{m[k}\,, \mathcal{D}_i B_{j]}{}^m\,\right] + \kappa_5\left[\mathcal{D}^m B_{[ij}\,, B_{k]m}\,\right] \\ &\quad \kappa_6\left[\mathcal{D}^m B_{m[k}\,, B_{ij]}\,\right] \end{aligned} \tag{6.14}$$

where the coefficients $\kappa_1, \ldots, \kappa_6$ on the r.h.s. remain to be fixed. The content of this equation as it stands, is thus the nonvanishing of the l.h.s., ensuring the existence of a field equation which involves the additional fields C_{ij} and Z_j.

It further remains to identify possible dynamical restrictions arising from the higher dimensional $SO(7)$ subrepresentations on this level. We leave this task for future work.

Since the dynamical equations (6.10), (6.13), and (6.14) depend essentially on a computation performed on the computer, at this point, some comments on the numerical credibility of these results are in order. The general form of the field equations is determined from group theory, the projection (4.31) merely determines the numerical prefactors; in particular, the existence of these equations depends exclusively on the non-vanishing of the relative factors on the l.h.s. of (6.10), (6.13), and (6.14), respectively. This procedure is certainly extremely sensitive to any kind of computational errors. However, we can at any point in the calculation check and verify the general algebraic relations (4.11), (4.14) and (4.15), as well as explicitly reproduce the multiplicities of subrepresentations collected in appendix B. This provides a sufficient number of checks to trust the numerical results, i.e. the existence of (6.10), (6.13), and (6.14).

6.3 Comments on the physics

The spectrum of $SO(2,1)$ singlets involves two pairs of fields (X_i, Z_i and B_{ij}, C_{ij}) with the same tensorial structure, but with different dimensions, since X_i, B_{ij}, and Z_i, C_{ij} have dimensions one and two, respectively. The field C seems to be associated with X. As a matter of fact, it is possible[6] to rewrite the field equations (6.10), (6.13), (6.14) solely in terms of $Y + C$. Then, if we reduce to four dimensions, there is a striking analogy with the case of electromagnetism in the presence of magnetic charge (see e.g. ref.[14]), where the field strength is build from two pieces, a homogeneous one such that the Bianchi identity gives zero, and another piece for which the Bianchi identity gives the magnetic current. Thus, we conclude that our dynamics in general involves magnetic charges.

On the other hand, the field B has the features of a two-form vector potential. An intriguing question is the role of the corresponding gauge transformations $B_{ij} \to B_{ij} + D_{[i}\Lambda_{j]} + \ldots$. However, the form of possible interactions with higher form gauge potentials appears to be highly restricted on general grounds (see [15] for a recent discussion). In the system studied here, the dynamical equation for B would be determined from the representation **21** at $p = 4$, which we have not yet fully investigated, since the calculation is rather delicate. For the moment, it is hence still possible that all terms vanish identically, such that there would be no field equation for B. Then B would be an arbitrary background field. Finally, the role of Z_i is unclear. It is somewhat surprising that it satisfies a first order differential equations. We leave all these questions to future studies.

[6]provided that κ_1 and κ_2 come out to be equal in (6.14), and after adding a term proportional to $D^j B_{ij}$ to Z_i,

7 Summary and Outlook.

In this paper we have studied the field content and the field equations implied by the weak dynamical constraint (3.11) as opposed to the usually employed strong dynamical constraint (2.9) which is known to be equivalent to the Yang-Mills equations of motion. The motivation for the weak constraint stemmed from the fact that it may be obtained as integrability condition of a Lax representation with scalar spectral parameter (3.10) which bears strong similarity with the Lax connection for selfdual four-dimensional Yang-Mills theory.

The underlying symmetry of the weak constraint is $SO(2,1) \times SO(7)$ rather than the original $SO(9,1)$. We have determined the physical field content of the theory in the space of $SO(2,1)$ singlets and shown that it restricts to the fields (5.26) appearing on the lowest two levels of the supercurl $M_{\alpha\beta}$. This naturally raises the question on the existence of closed recurrence relations, which, in analogy to (5.1), would explicitly determine the higher superfield levels of the supercurl. In fact, by manipulating the Bianchi identities together with (3.11), one may obtain superfield equations which in their lowest component reproduce (6.4). In particular, the new fields C_{ij} and Z_i appear as level zero components of additional superfields $\widetilde{C}_{ij}$ and $\widetilde{Z}_i$, respectively. Complementing these equations by similar relations for the superfields $\widetilde{C}_{ij}$, $\widetilde{Z}_i$, one arrives at a system of recurrence relations which determine the superfield $M_{\alpha\beta}$ entirely in terms of the physical fields (5.26) from the lowest orders. The dynamical equations (6.10), (6.13) and (6.14), implied by the weak superspace constraint, will then be likewise promoted from component equations to superfield equations. Details remain to be worked out. More importantly, it remains to actually construct explicit solutions of the Lax equations (3.10) with the required form of the dependence upon the spectral parameter (equation (3.17) and (3.19), respectively).

The above analysis of superfield components and dynamical equations has been restricted to the system after dimensional reduction to seven dimensions. In this case, we could consistently truncate the field content to $SO(2,1)$ singlets which essentially simplified the situation. In principle, the same method can be applied including the higher $SO(2,1)$ spin fields, though the calculations become far more tedious. We mention here some of the novel features which may be discovered already at level $p=4$.

First, the dynamical equations (6.10), (6.13), (6.14), obtained in the reduced situation, receive additional contributions due to the action of the $(\mathbf{3},\mathbf{1})$ part of the operator T in (4.31). In particular, the completion of equation (6.13) will at least contain terms $\mathcal{D}^8 F_{i8}$, $\mathcal{D}^{\pm} F_{i\pm}$ since obviously the weak superspace constraint (4.28) cannot yield equations stronger than the original Yang-Mills equations of motion (6.6).

In addition, the original Yang-Mills equations of motion now contain a part transforming in the $(\mathbf{3},\mathbf{1})$ which should be recovered from the corresponding sector at level $p=4$. However, here an essential qualitative difference arises: the space $\mathcal{M}^{-\,[4]}_{\perp\,(\mathbf{3},\mathbf{1})}$ is empty. Thus, with the weak superspace constraint (3.11) this part of the original dynamical equations disappears and we go truly off-shell. It may, however, not be excluded that further dynamical restrictions arise at the higher levels in the supercurl $M_{\alpha\beta}$.

Finally, we discover that for the full system, the superfield level $p=4$ is not entirely determined by the lower levels. In particular, we find new independent fields arising at this level in the $(\mathbf{5},\mathbf{7})$ and $(\mathbf{5},\mathbf{21})$ of $SO(2,1)\times SO(7)$, i.e. with $SO(2,1)$ spin 2. To obtain closed recurrence relations for the supercurl $M_{\alpha\beta}$ one would thus be forced to include at least the next level $p=6$ in the analysis. Likewise, the dynamical field content will essentially depend on the higher levels. We leave this for future work.

Another point we have omitted so far, is the fermionic spectrum of the theory which may be analyzed with exactly the same methods that have been presented here for the bosonic sector. In particular, the (possibly broken) supersymmetry should help to better understand the nature of the underlying physical system.

We finally mention the possibility to recover in this framework and upon dimensional reduction some of the classical higher spin gauge theories, which have been constructed by Vasiliev (see e.g. [16] for a review) and recently [17] been brought into the context of a possible M-theoretic origin.

Acknowledgements
This work was done in part while one of us (J.-L. G.) was visiting the Physics Department of the University of California at Los Angeles. He is grateful for the warm hospitality, and generous financial support extended to him. The work of H. S. was supported by EU contract ERBFMRX-CT96-0012. Discussions with E. Cremmer, E. D'Hoker, P. Forgacs and K. Stelle have been very useful.

Appendix

A The $O(9,1)$ characters.

In this appendix, we compute the characters of the reducible representations which appear in section 4 . The path followed is similar to the calculation of string characters of ref.[13], and we shall refer to that paper for details. In general, the $O(9,1)$ characters are defined as $\chi(\vec{v}) = \text{ Tr } \left(e^{\sum_i v_i \mathcal{H}_i}\right)$, where v_i are arbitrary parameters, where the trace is taken in the representation considered, and $\mathcal{H}_i$, $i=1,\ldots,5$ are a set of five commuting elements of the Lie algebra. Using a parametrization analogous to the one used in ref.[13] for $O(8)$ spinors, one easily sees for instance that the character of the **16** representation[7] is given by

$$\chi_{\mathbf{16}}(\vec{v}) = \sum_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1 \\ \text{odd } \# = 1}} \prod_i e^{\frac{1}{2} v_i \epsilon_i} \ . \tag{A.1}$$

A.1 The unconstrained character.

In this subsection, we first compute the character associated with the representation span by the full $M_{\alpha\beta}$, by considering the trace over the full space $\mathcal{M}$. The $\alpha\beta$ indices then contribute a factor

$$\chi_{\underbrace{\mathbf{16}\otimes\mathbf{16}}_{s}}(\vec{v}) = \frac{1}{2}\left\{\left[\sum_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1 \\ \text{odd } \# = 1}} \prod_i e^{\frac{1}{2} v_i \epsilon_i}\right]^2 + \sum_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1 \\ \text{odd } \# = 1}} \prod_i e^{\frac{1}{2} 2 v_i \epsilon_i}\right\}$$

Concerning the θ part, one works in an occupation number basis where $N_\alpha = \theta^\alpha \partial_\alpha$ is simultaneously diagonal for $\alpha = 1,\cdots,16$. Since the Lie group generators commute with the grading operator $\mathcal{R}$, it is convenient to introduce in general characters of the type $\chi(\vec{v}|q) = \text{ Tr } \left(q^{\mathcal{R}} e^{\sum_i v_i \mathcal{H}_i}\right)$. Then the calculations becomes identical to a part of the string calculation, where the role of $\mathcal{R}$ is played by the Virasoro generator L_0. Altogether, one finds

[7] In this appendix boldface dimensions refer to $O(9,1)$ representations

that the character without any constraint denoted χ_u is given by

$$\chi_u(\vec{v}|q) = \frac{1}{2}\left\{\left[\sum_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1\\ \text{odd }\#=1}} \prod_i e^{\frac{1}{2}v_i\epsilon_i}\right]^2 + \sum_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1\\ \text{odd }\#=1}} \prod_i e^{\frac{1}{2}2v_i\epsilon_i}\right\} \times$$

$$\times \prod_{\substack{\epsilon_1,\cdots,\epsilon_5=\pm 1\\ \text{odd }\#=1}} \left(1 + q\prod_i e^{\frac{1}{2}v_i\epsilon_i}\right) \tag{A.2}$$

A.2 The character corresponding to $\mathcal{M}^{\pm}$.

To determine them, we first compute the character with S introduced. This is straightforward since S is a group invariant. Using again the occupation numbers operators $N_\alpha = \theta^\alpha\partial_\alpha$, one may verify that

$$\chi_S(\vec{v}|q) \equiv \text{ Tr } \left(e^{\sum_{i=1}^5 v_i\mathcal{H}_i} S\right)$$

$$= \underbrace{\sum_{\alpha,\beta}}_{s} \text{ Tr}_{\,\theta} \left\{ q^{\sum_\gamma N_\gamma} \prod_i \left(e^{v_i\mathcal{H}_i}\right)_{\alpha,\beta;\alpha,\beta} e^{v_i\sum_\rho(\mathcal{H}_i)^\rho{}_\rho N_\rho} \left(N_\alpha + N_\beta\right)\right\} \tag{A.3}$$

where the trace over $\alpha\beta$ only involves the symmetric states. After some straightforward computation on finds that

$$2\chi_u(\vec{v}|q) - \chi_S(\vec{v}|q) = \sum_{\{\epsilon\}\{\epsilon'\}} \prod_i e^{\frac{1}{2}v_i\left(\epsilon_i+\epsilon'_i\right)} \prod_{\{\eta\}\neq\{\epsilon\},} \left(1 + q\prod_i e^{\frac{1}{2}v_i\eta_i}\right)$$

$$+\sum_{\{\epsilon\}} \prod_i e^{v_i(\epsilon_i)} \prod_{\{\eta\}\neq\{\epsilon\}} \left(1 + q\prod_i e^{\frac{1}{2}v_i\eta_i}\right) \tag{A.4}$$

Clearly,

$$2\chi_u - \chi_S \equiv \text{ Tr } \left((S-2)\, q^{\mathcal{R}} \prod_i e^{v_i\mathcal{H}_i}\right) = \text{ Tr}_{\,\mathcal{M}^+} \left((\mathcal{R}+2)\, q^{\mathcal{R}} \prod_i e^{v_i\mathcal{H}_i}\right).$$

An easy computation then gives

$$\chi^{+}\left(\vec{v}|q\right) \equiv \ \mathrm{Tr}_{\mathcal{M}^{+}}\left(q^{\mathcal{R}}\prod_{i} e^{v_i \mathcal{H}_i}\right) = q^{-2}\int_0^q dxx\left\{2\chi_u\left(\vec{v}|x\right) - \chi_S\left(\vec{v}|x\right)\right\} \tag{A.5}$$

This gives

$$\begin{aligned}\chi^{+}\left(\vec{v}|q\right)) = \sum_{p=0}^{14} q^p\left(p+1\right) \sum_{\{\eta_1\}<\{\eta_2\}<\cdots<\{\eta_{p+2}\}} e^{\frac{1}{2}\vec{v}\cdot\left(\sum_{r=1}^{p+2}\vec{\eta}_r\right)} \\ + \sum_{p=0}^{15} q^p \sum_{\{\epsilon\}\neq\{\eta_1\}<\{\eta_2\}<\cdots<\{\eta_p\}} e^{\frac{1}{2}\vec{v}\cdot\left(\sum_{r=1}^{p}\vec{\eta}_r + 2\vec{\epsilon}\right)},\end{aligned} \tag{A.6}$$

where all summations over η and ϵ are understood with an odd total number of positive sign, respectively. Of course the dimensions are derived from the particular case where $\vec{v}=0$. One may verify in this way formulae given in the main text, as well as derive others. In particular,

$$\dim \mathcal{M}^{+} = 16\left(2^{17}-1\right) - 17\left(2^{16}-1\right) = 983041 \tag{A.7}$$

It would be interesting to derive the characters $\chi_{\|}^{+}$, and $\chi_{\perp}^{-}$ which determine the physical field content, and the set of field equations. This is much harder, since K breaks the $O(9,1)$ invariance. It is left for further studies.

B Group decomposition of superfields

In this appendix we give the decomposition of the lowest levels of the space of superfields

$$\mathcal{M} \ = \ \mathcal{M}_{\|} + \mathcal{M}_{\perp} \ = \ \mathcal{M}^{+} + \mathcal{M}^{-}\ , \tag{B.1}$$

with respect to $SO(9,1)$ and $SO(2,1)\times SO(7)$, respectively.

At level $p=2$ with $\mathcal{M}_{\|}, \mathcal{M}_{\perp}$ defined by the strong constraint (4.24) we find the following $SO(9,1)$ multiplicities

$SO(9,1)$ mult.	**45**	**210**	**945**	**1050**	**5940**	**6930**
$\mathcal{M}$	2	2	2	1	1	1
$\mathcal{M}_{\|}$	1	1	1			
$\mathcal{M}_{\perp}$	1	1	1	1	1	1
$\mathcal{M}^{+}$	1	1	1		1	
$\mathcal{M}^{-}$	1	1	1	1		1

(B.2)

For the weak constraint (4.28) this table takes the form – now in terms of representations of $SO(2,1) \times SO(7)$

$\mathcal{M}$	$4 \cdot (\mathbf{3},\mathbf{1})$	$2 \cdot (\mathbf{5},\mathbf{7})$ $5 \cdot (\mathbf{3},\mathbf{7})$ $4 \cdot (\mathbf{1},\mathbf{7})$	$3 \cdot (\mathbf{5},\mathbf{21})$ $6 \cdot (\mathbf{3},\mathbf{21})$ $6 \cdot (\mathbf{1},\mathbf{21})$	$3 \cdot (\mathbf{3},\mathbf{27})$	$2 \cdot (\mathbf{5},\mathbf{35})$ $8 \cdot (\mathbf{3},\mathbf{35})$ $4 \cdot (\mathbf{1},\mathbf{35})$	...
$\mathcal{M}_\parallel$	$3 \cdot (\mathbf{3},\mathbf{1})$	$1 \cdot (\mathbf{5},\mathbf{7})$ $3 \cdot (\mathbf{3},\mathbf{7})$ $3 \cdot (\mathbf{1},\mathbf{7})$	$1 \cdot (\mathbf{5},\mathbf{21})$ $3 \cdot (\mathbf{3},\mathbf{21})$ $4 \cdot (\mathbf{1},\mathbf{21})$	$2 \cdot (\mathbf{3},\mathbf{27})$	$4 \cdot (\mathbf{3},\mathbf{35})$ $2 \cdot (\mathbf{1},\mathbf{35})$	...
$\mathcal{M}_\perp$	$1 \cdot (\mathbf{3},\mathbf{1})$	$1 \cdot (\mathbf{5},\mathbf{7})$ $2 \cdot (\mathbf{3},\mathbf{7})$ $1 \cdot (\mathbf{1},\mathbf{7})$	$2 \cdot (\mathbf{5},\mathbf{21})$ $3 \cdot (\mathbf{3},\mathbf{21})$ $2 \cdot (\mathbf{1},\mathbf{21})$	$1 \cdot (\mathbf{3},\mathbf{27})$	$2 \cdot (\mathbf{5},\mathbf{35})$ $4 \cdot (\mathbf{3},\mathbf{35})$ $2 \cdot (\mathbf{1},\mathbf{35})$	...
$\mathcal{M}^+$	$2 \cdot (\mathbf{3},\mathbf{1})$	$1 \cdot (\mathbf{5},\mathbf{7})$ $3 \cdot (\mathbf{3},\mathbf{7})$ $2 \cdot (\mathbf{1},\mathbf{7})$	$2 \cdot (\mathbf{5},\mathbf{21})$ $3 \cdot (\mathbf{3},\mathbf{21})$ $3 \cdot (\mathbf{1},\mathbf{21})$	$2 \cdot (\mathbf{3},\mathbf{27})$	$1 \cdot (\mathbf{5},\mathbf{35})$ $4 \cdot (\mathbf{3},\mathbf{35})$ $2 \cdot (\mathbf{1},\mathbf{35})$	...
$\mathcal{M}^-$	$2 \cdot (\mathbf{3},\mathbf{1})$	$1 \cdot (\mathbf{5},\mathbf{7})$ $2 \cdot (\mathbf{3},\mathbf{7})$ $2 \cdot (\mathbf{1},\mathbf{7})$	$1 \cdot (\mathbf{5},\mathbf{21})$ $3 \cdot (\mathbf{3},\mathbf{21})$ $3 \cdot (\mathbf{1},\mathbf{21})$	$1 \cdot (\mathbf{3},\mathbf{27})$	$1 \cdot (\mathbf{5},\mathbf{35})$ $4 \cdot (\mathbf{3},\mathbf{35})$ $2 \cdot (\mathbf{1},\mathbf{35})$	...

(B.3)

The multiplicities of $\mathcal{M}_\parallel$, $\mathcal{M}_\perp$ here are determined by computing the decomposition of tensor products

$$(\mathbf{8} \times \mathbf{8})_s \times (\mathbf{8} \times \mathbf{8} \times \ldots \times \mathbf{8})_a \times (\mathbf{8} \times \mathbf{8} \times \ldots \times \mathbf{8})_a \,, \qquad \text{etc.} \;;$$

To obtain the multiplicities of $\mathcal{M}^\pm$ one needs in addition the decomposition of the Young tableaux (4.13).

At level $p = 4$ we find with the strong constraint (4.24) the following multiplicities of the lowest dimensional $SO(9,1)$ representations

$SO(9,1)$ mult.	$\mathbf{10}$	$\mathbf{120}$	$\mathbf{126}$	$\overline{\mathbf{126}}$	$\mathbf{320}$	...
$\mathcal{M}$	1	1	1	2	2	...
$\mathcal{M}_\parallel$				1	1	...
$\mathcal{M}_\perp$	1	1	1	1	1	...
$\mathcal{M}^+$				1	1	...
$\mathcal{M}^-$	1	1	1	1	1	...

(B.4)

whereas the weak constraint (4.28) implies the decomposition

$\mathcal{M}$	$2\cdot(\mathbf{7},\mathbf{1})$ $4\cdot(\mathbf{5},\mathbf{1})$ $8\cdot(\mathbf{3},\mathbf{1})$ $1\cdot(\mathbf{1},\mathbf{1})$	$2\cdot(\mathbf{7},\mathbf{7})$ $11\cdot(\mathbf{5},\mathbf{7})$ $13\cdot(\mathbf{3},\mathbf{7})$ $9\cdot(\mathbf{1},\mathbf{7})$	$2\cdot(\mathbf{7},\mathbf{21})$ $15\cdot(\mathbf{5},\mathbf{21})$ $19\cdot(\mathbf{3},\mathbf{21})$ $14\cdot(\mathbf{1},\mathbf{21})$	$2\cdot(\mathbf{7},\mathbf{27})$ $7\cdot(\mathbf{5},\mathbf{27})$ $13\cdot(\mathbf{3},\mathbf{27})$ $5\cdot(\mathbf{1},\mathbf{27})$	$5\cdot(\mathbf{7},\mathbf{35})$ $15\cdot(\mathbf{5},\mathbf{35})$ $28\cdot(\mathbf{3},\mathbf{35})$ $10\cdot(\mathbf{1},\mathbf{35})$	$\ldots$
$\mathcal{M}_\parallel$	$1\cdot(\mathbf{7},\mathbf{1})$ $2\cdot(\mathbf{5},\mathbf{1})$ $5\cdot(\mathbf{3},\mathbf{1})$	$1\cdot(\mathbf{7},\mathbf{7})$ $6\cdot(\mathbf{5},\mathbf{7})$ $7\cdot(\mathbf{3},\mathbf{7})$ $5\cdot(\mathbf{1},\mathbf{7})$	$7\cdot(\mathbf{5},\mathbf{21})$ $9\cdot(\mathbf{3},\mathbf{21})$ $8\cdot(\mathbf{1},\mathbf{21})$	$1\cdot(\mathbf{7},\mathbf{27})$ $3\cdot(\mathbf{5},\mathbf{27})$ $7\cdot(\mathbf{3},\mathbf{27})$ $2\cdot(\mathbf{1},\mathbf{27})$	$1\cdot(\mathbf{7},\mathbf{35})$ $5\cdot(\mathbf{5},\mathbf{35})$ $13\cdot(\mathbf{3},\mathbf{35})$ $4\cdot(\mathbf{1},\mathbf{35})$	$\ldots$
$\mathcal{M}_\perp$	$1\cdot(\mathbf{7},\mathbf{1})$ $2\cdot(\mathbf{5},\mathbf{1})$ $3\cdot(\mathbf{3},\mathbf{1})$ $1\cdot(\mathbf{1},\mathbf{1})$	$1\cdot(\mathbf{7},\mathbf{7})$ $5\cdot(\mathbf{5},\mathbf{7})$ $6\cdot(\mathbf{3},\mathbf{7})$ $4\cdot(\mathbf{1},\mathbf{7})$	$2\cdot(\mathbf{7},\mathbf{21})$ $8\cdot(\mathbf{5},\mathbf{21})$ $9\cdot(\mathbf{3},\mathbf{21})$ $6\cdot(\mathbf{1},\mathbf{21})$	$1\cdot(\mathbf{7},\mathbf{27})$ $4\cdot(\mathbf{5},\mathbf{27})$ $6\cdot(\mathbf{3},\mathbf{27})$ $3\cdot(\mathbf{1},\mathbf{27})$	$4\cdot(\mathbf{7},\mathbf{35})$ $10\cdot(\mathbf{5},\mathbf{35})$ $15\cdot(\mathbf{3},\mathbf{35})$ $6\cdot(\mathbf{1},\mathbf{35})$	$\ldots$
$\mathcal{M}^+$	$1\cdot(\mathbf{7},\mathbf{1})$ $2\cdot(\mathbf{5},\mathbf{1})$ $3\cdot(\mathbf{3},\mathbf{1})$	$1\cdot(\mathbf{7},\mathbf{7})$ $5\cdot(\mathbf{5},\mathbf{7})$ $5\cdot(\mathbf{3},\mathbf{7})$ $3\cdot(\mathbf{1},\mathbf{7})$	$1\cdot(\mathbf{7},\mathbf{21})$ $6\cdot(\mathbf{5},\mathbf{21})$ $7\cdot(\mathbf{3},\mathbf{21})$ $5\cdot(\mathbf{1},\mathbf{21})$	$1\cdot(\mathbf{7},\mathbf{27})$ $3\cdot(\mathbf{5},\mathbf{27})$ $5\cdot(\mathbf{3},\mathbf{27})$ $2\cdot(\mathbf{1},\mathbf{27})$	$2\cdot(\mathbf{7},\mathbf{35})$ $6\cdot(\mathbf{5},\mathbf{35})$ $10\cdot(\mathbf{3},\mathbf{35})$ $3\cdot(\mathbf{1},\mathbf{35})$	$\ldots$
$\mathcal{M}^-$	$1\cdot(\mathbf{7},\mathbf{1})$ $2\cdot(\mathbf{5},\mathbf{1})$ $5\cdot(\mathbf{3},\mathbf{1})$ $1\cdot(\mathbf{1},\mathbf{1})$	$1\cdot(\mathbf{7},\mathbf{7})$ $6\cdot(\mathbf{5},\mathbf{7})$ $8\cdot(\mathbf{3},\mathbf{7})$ $6\cdot(\mathbf{1},\mathbf{7})$	$1\cdot(\mathbf{7},\mathbf{21})$ $9\cdot(\mathbf{5},\mathbf{21})$ $12\cdot(\mathbf{3},\mathbf{21})$ $9\cdot(\mathbf{1},\mathbf{21})$	$1\cdot(\mathbf{7},\mathbf{27})$ $4\cdot(\mathbf{5},\mathbf{27})$ $8\cdot(\mathbf{3},\mathbf{27})$ $3\cdot(\mathbf{1},\mathbf{27})$	$3\cdot(\mathbf{7},\mathbf{35})$ $9\cdot(\mathbf{5},\mathbf{35})$ $18\cdot(\mathbf{3},\mathbf{35})$ $7\cdot(\mathbf{1},\mathbf{35})$	$\ldots$

(B.5)

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